MAT157Y term test 1, October 2018

Please answer all the questions. You can answer the questions in any order, but please use separate pages for different questions, and mark the question number at the top corner of the page.

The five questions have equal weight. Clarity counts. Simplify your solutions when possible. A response "I don't know" and nothing more, to any question or section of a question, will give you 20% of the points for that question or section. You have 100 minutes. No aids allowed. The list of axioms for the real numbers is attached after the questions.

(1) For each of the following statements, determine if it is true or false, and justify your answer briefly, in one sentence.

 (a) The set of integers, with its usual operations of addition and multiplication, satisfies the arithmetic axioms (P1)–(P9).

 (b) For every $\epsilon > 0$ there is a natural number m such that $\frac{1}{m} < \epsilon$.

 (c) If a function f is continuous at the point 5 and $\lim\limits_{x\to 5} f(x) = 17$, then $f(5) = 17$.

 (d) If $f\colon \mathbb{R} \to \mathbb{R}$ is a function, $f(x)$ is positive for all rational x, and $f(x)$ is negative for all irrational x, then f is not continuous at 0.

(2) (a) Does the set $\{x \mid x^2 - x - 12 < 0\}$ have a greatest lower bound (infimum)? If so, find it. If not, explain in one sentence.

 (b) Let A be a set of real numbers. Prove that if y is a least upper bound (supremum) for the set $-A := \{-x \mid x \in A\}$ then $-y$ is a greatest lower bound for the set A.

(3) (a) Let f be a function. Suppose that $f(x+h) - f(x) = a(x)b(h) + c(x)d(h)$ for all x and h where a, b, c, d are functions with the following properties: $|a(x)| \leq 1$ and $|c(x)| \leq 1$ for all x, $b(h)$ and $d(h)$ are continuous at $h = 0$, and $b(0) = d(0) = 0$. Show that f is continuous everywhere.

 (b) Assuming that the sine and cosine functions are continuous at 0, use trigonometric identities and Part (a) to prove that the sine function is continuous everywhere.

(4) (a) Sketch the graph of a function $f\colon \mathbb{R} \to \mathbb{R}$ that has all the following properties. f is an odd function (i.e. $f(-x) = -f(x)$ for all x); $f(1) = 1$; $x \mapsto f(x)$ is continuous at all $x \neq 0$; $\lim\limits_{x\to\infty} f(x) = 0$; $\lim\limits_{\substack{x\to 0\\ x>0}} f(x) = 0.5$.

 (b) To the best of your understanding (and without using derivatives), sketch the graph of the function $t \mapsto \frac{1-t^2}{1+t^2}$.

(5) Find a number c such that the function

$$f_c(h) := \begin{cases} \dfrac{\sqrt{5+h} - \sqrt{5}}{h} & h \neq 0 \\ c & h = 0 \end{cases}$$

is continuous at $h = 0$. Show your work, but do not explain.

What is the domain of this function?

Postulates for the real numbers

(P1) (Associative law for addition) $a + (b + c) = (a + b) + c.$

(P2) (Existence of an additive identity) $a + 0 = 0 + a = a.$

(P3) (Existence of additive inverses) $a + (-a) = (-a) + a = 0.$

(P4) (Commutative law for addition) $a + b = b + a.$

(P5) (Associative law for multiplication) $a \cdot (b \cdot c) = (a \cdot b) \cdot c.$

(P6) (Existence of a multiplicative identity) $a \cdot 1 = 1 \cdot a = a; \quad 1 \neq 0.$

(P7) (Existence of multiplicative inverses) $a \cdot a^{-1} = a^{-1} \cdot a = 1, \text{ for } a \neq 0.$

(P8) (Commutative law for multiplication) $a \cdot b = b \cdot a.$

(P9) (Distributive law) $a \cdot (b + c) = a \cdot b + a \cdot c.$

(P10) (Trichotomy law) For every number a, one and only one of the following holds:

(i) $a = 0$,
(ii) a is in the collection P,
(iii) $-a$ is in the collection P.

(P11) (Closure under addition) If a and b are in P, then $a + b$ is in P.

(P12) (Closure under multiplication) If a and b are in P, then $a \cdot b$ is in P.

A number x is a **least upper bound** of A if

(1) x is an upper bound of A,
and (2) if y is an upper bound of A, then $x \leq y$.

A set A of real numbers is **bounded above** if there is a number x such that

$$x \geq a \quad \text{for every } a \text{ in } A.$$

Such a number x is called an **upper bound** for A.

(P13) (The least upper bound property) If A is a set of real numbers, $A \neq \emptyset$, and A is bounded above, then A has a least upper bound.

from: Spivak, "Calculus"