MAT157Y term test 1, October 2020

Please answer all the questions. Submit your answers on Crowdmark.

The five questions have equal weight. Clarity counts. Simplify your solutions when possible.

A response "I don't know" and nothing more, to any question or section of a question, will give you 20% of the points for that question or section.

The list of axioms for the real numbers is attached after the questions.

You have 100 consecutive minutes to write your solutions. After 100 minutes, stop writing. You will then have 10 additional minutes to upload your solutions on Crowdmark.

No aids allowed. (Nothing! No physical nor electronic note(s) or book(s); no access to any online materials; no calculator; no verbal nor written communication with anyone other than Yael Karshon or David Miyamoto.)

After the completion of the test, we will ask you to report on any abnormalities that may have occurred in connection with the test that might be interpreted as a suspected academic offence. If relevant, we would then schedule an individual appointment to discuss what happened and see whether and how to proceed.

I also remain available for individual appointments regarding any concern that you might have about the course.

(1) For each of the following statements, determine if it is true or false. Add a one-sentence explanation. Your explanation does not need to be a complete proof.

(a) "The non-existence of an x such that $x \cdot x + 1 = 0$ can be deduced from the arithmetic axioms."

(b) "The set of natural numbers is not bounded from above."

(c) "There is a $\delta > 0$ such that, for all $a \neq 0$, if $|x - a| < \delta$, then $|\frac{1}{x} - \frac{1}{a}| < \frac{1}{10}$."

(d) "If $\lim_{x \to 1} f(x) = 5$, and $\lim_{x \to 1} g(x) = \ell$, and $f(x) \neq g(x)$ for all x, then $\ell \neq 5$."

(2) Let A be a set of real numbers such that $\inf A = -1$ and $\sup A = 1$.

For each of the statements below, either declare that it is necessarily true and prove it, or declare that can be false and give a counterexample.

(a) "The set A has a maximal element."

(b) "The set A contains some positive number."

(c) "The number 0 is in A."

(d) "The set of lower bounds for A is bounded from above by 0."

(3) For each natural number n, consider the real number $a_n := 4.\underbrace{99\ldots9}_{n \text{ times}}$.

Prove that the supremum of the set $\{a_n \mid n \in \mathbb{N}\}$ is 5.

(4) Let $f \colon \mathbb{R} \to \mathbb{R}$ be a function, and suppose that $\lim_{x \to 0} \dfrac{f(x)}{x} = 4$. For each of the following expressions, determine the value of the limit, or declare that the limit does not exist, or declare that you do not have enough information. If the limit does not exist but the function diverges to ∞ or to $-\infty$, express this by writing "$\ldots = \infty$" or "$\ldots = -\infty$". Do not justify.

(a) $\lim_{x \to \infty} x f\big(\frac{1}{x}\big)$ (b) $\lim_{x \to 0} \dfrac{x}{f(x)}$ (c) $\lim_{x \to 0} f(x)$ (d) $\lim_{x \to 0} \dfrac{f(x)}{x^2}$

(e) $\lim_{x \to 0} \big(\dfrac{f(x)}{x}\big)^2$ (f) $\lim_{x \to 0} \dfrac{|f(x)|}{x}$ (g) $\lim_{x \to 0} \dfrac{f(x)}{x^2 + 3x}$ (h) $\lim_{x \to 0} \dfrac{f(x^2)}{x^2}$

(5)

(a) Find a function whose graph is the set $\{(x, y) \mid |x| + |y| = 1\}$, or explain why such a function does not exist.

(b) Give an example of a function that is one-to-one and is not monotone, or explain why such a function does not exist.

(c) Give an example of a function f that is everywhere positive and that satisfies $\lim_{x \to 1} f(x) = 0$, or explain why such a function does not exist.

(d) Give an example of a continuous function f that is everywhere positive and that satisfies $\lim_{x \to 1} f(x) = 0$, or explain why such a function does not exist.

THE REAL NUMBERS

Structure:

A set $\mathbb{R}$; two distinguished elements $\mathbf{0}$ and $\mathbf{1}$ of $\mathbb{R}$; an **addition** operation; a **multiplication** operation; a distinguished subset $\boldsymbol{P}$ of $\mathbb{R}$.

Arithmetic axioms:

Addition is associative: for all $a, b, c \in \mathbb{R}$, $a + (b + c) = (a + b) + c$

Neutral element for addition: for all $a \in \mathbb{R}$, $a + 0 = 0 + a = a$

Additive inverse: for every $a \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $a + x = x + a = 0$

Addition is commutative: for all $a, b \in \mathbb{R}$, $a + b = b + a$

Multiplication is associative: for all $a, b, c \in \mathbb{R}$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

Neutral element for multiplication: $1 \neq 0$, and, for all $a \in \mathbb{R}$, $a \cdot 1 = 1 \cdot a = a$

Multiplicative inverse: for every $a \neq 0$ there exists an x such that $a \cdot x = x \cdot a = 1$

Multiplication is commutative: for all $a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$

Distributive law: for all $a, b, c \in \mathbb{R}$, $a \cdot (b + c) = a \cdot b + a \cdot c$

Order axioms:

Trichotomy: for every $x \in \mathbb{R}$, exactly one of $(x = 0;\ x \in P;\ -x \in P)$ holds

P closed under addition: for every $x, y \in \mathbb{R}$, if $x, y \in P$, then $x + y \in P$

P closed under multiplication: for every $x, y \in \mathbb{R}$, if $x, y \in P$, then $x \cdot y \in P$

An **upper bound** for a set of real numbers X is a number u such that $x \leq u$ for all $x \in X$. The set X is **bounded from above** if it has an upper bound. The set X is **bounded** if it is bounded from above and from below. A **least upper bound** of X is a number u such that u is an upper bound for X and such that, for every upper bound y for X, $u \leq y$.

Least upper bound axiom:

Every set of real numbers that is non-empty and bounded from above has a least upper bound