MAT157Y term test 2, December 2018

Please answer all the questions; the five questions have equal weight. Clarity counts. Simplify your solutions when possible. A response "I don't know" and nothing more, to any question or section of a question, will give you 20% of the points for that question or section. You can answer the questions in any order, but please use separate pages for different questions, and mark the question number at the top corner of the page. You have 100 minutes. No aids allowed.

(1) For each of the following statements, determine if it is true or false. Include an explanatory sentence or diagram or formula. You do not need to give a complete proof.

(a) The function $f(x) := \begin{cases} x^2\cos(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at $x = 0$.

(b) For any continuous function $f\colon \mathbb{R} \to \mathbb{R}$, if $f(x) = x^2$ whenever $|x| > 5$, then there exists λ such that $f(\lambda) \leq f(x)$ for all x.

(c) For any continuous function $f\colon \mathbb{R} \to \mathbb{R}$, if $f(x) > 0$ whenever $|x| > 5$, then there exists λ such that $f(\lambda) \leq f(x)$ for all x.

(d) For any function $f\colon \mathbb{R} \to \mathbb{R}$, if f is differentiable and $f'(x) > 0$ for all x, then there does not exist a λ such that $f(\lambda) \geq f(x)$ for all $x \in \mathbb{R}$.

(2) Let $f(x) = x^7 + 2x - 5$.

(a) Prove that f is one-to-one.

(b) What is the image of f? Justify in 1–2 sentences; do not give a detailed proof.

(c) Let f^{-1} be the inverse function. Prove that $(f^{-1})'(t) \leq \frac{1}{2}$ for all t.

(3) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that $f(0) = 0$ and that $|f'(x)| < \frac{1}{2}$ for all x. Prove that $f(10) < 5$.

(4) Find the maximum and minimum values of the function $f(x) = \frac{x}{x^2+1}$ on the interval $[0, 2]$. Do not explain, but show your work.

(5) $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function. Define a new function $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) := \begin{cases} (f(x))^2 & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) \leq 0 \end{cases}$$

Let c be a point in $\mathbb{R}$.

(a) Prove that if $f(c) > 0$ then g is differentiable at c.

(b) Prove that if $f(c) = 0$ then g is differentiable at c.