MAT157Y TERM TEST 2, DECEMBER 2020

Please submit the declaration that is labeled as Question 0.

If anything happened or will happen today that might be interpreted as a suspected academic offence, by yourself or by another student, please ask for an individual appointment with Yael, to discuss what happened and whether and how to proceed.

Please answer Questions 1–5. Submit your answers on Crowdmark. These questions have equal weight.

Question 6 is optional and has zero weight.

A response "I don't know" and nothing more, to any question or section of a question, will give you 20% of the points for that question or section.

Clarity counts. Simplify your solutions when possible.

You have 100 consecutive minutes to write your solutions. After 100 minutes, stop writing. You will then have 10 additional minutes to upload your solutions on Crowdmark.

No aids allowed. (Nothing! No physical nor electronic note(s) or book(s); no access to any online materials; no calculator; no verbal nor written communication with anyone other than Yael Karshon or David Miyamoto.)

Yael and David also remain available for individual appointments regarding any concern that you might have about the course.

(0) Please copy the following declaration and sign your name below it:

> I confirm that I wrote this test, or will write this test, entirely on my own, without using any aids or assistance, and without communicating with anyone about the content of the test between 9am and 9pm today.

(1) For each of the following statements, determine if it is true or false. Add an explanatory sentence or diagram or formula. You do not need to give a complete proof.

(a) If $f\colon \mathbb{R} \to \mathbb{R}$ has a strict local minimum at 0 and $f''(0)$ exists, then $f''(0) > 0$.

(b) If $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and satisfies $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to-\infty} f(x) = \infty$, then there exists a point x where $f'(x) = 0$.

(c) If $f\colon [0,\infty) \to \mathbb{R}$ is continuous on $[0,\infty)$, differentiable on $(0,\infty)$, and satisfies $\lim\limits_{x\to\infty} f'(x) = 0$, then f is bounded.

(d) If $f\colon \mathbb{R} \to \mathbb{R}$ is a strictly convex function and it has a local minimum at 0, then it has a strict local minimum at 0.

(2)

(a) Calculate the maximum value of $\dfrac{x^2+x+1}{x^2+1}$ for $x \in \mathbb{R}$.

(b) Calculate $f'(0)$, where $f(x) = \begin{cases} \frac{x}{1-e^{-x}} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Justify your work, briefly.

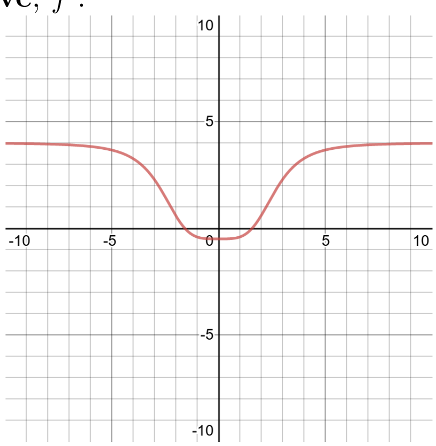
(3)

(a) Suppose that f is differentiable at 0 and $f(0) = 0$ and that g is continuous at 0. Let $h(x) := f(x)g(x)$. Prove that h is differentiable at 0.

(b) Suppose that f is differentiable at 0 and that $|f(\alpha) - f(\beta)| \leq (\alpha - \beta)^2$ for all α and β. What can you say about $f'(0)$?

(4) Let f be a continuous function on the interval $[-2,2]$. Prove that there exists a point on the graph of f whose distance to the origin is smallest among all the points on the graph of f.

(5) Here is a graph of a differentiable function f. Sketch a rough graph of its **derivative**, f'.



(6) (Optional; no credit:) Write a short one-paragraph response to the attached text, "The Lever of Mahomet".

4 The Lever of Mahomet

By RICHARD COURANT
and HERBERT ROBBINS

SUPPOSE a train travels in a finite time from station A to station B along a straight section of track. The journey need not be of uniform speed or acceleration. The train may act in any manner, speeding up, slowing down, coming to a halt, or even backing up for a while, before reaching B. But the exact motion of the train is supposed to be known in advance; that is, the function $s = f(t)$ is given, where s is the distance of the train from station A, and t is the time, measured from the instant of departure. On the floor of one of the cars a rod is pivoted so that it may move without friction either forward or backward until it touches the floor. (If it does touch the floor, we assume that it remains on the floor henceforth; this will be the case if the rod does not bounce.) *We ask if it is possible to place the rod in such a position that if it is released at the instant when the train starts and allowed to move solely under the influence of gravity and the motion of the train, it will not fall to the floor during the entire journey from* A *to* B.

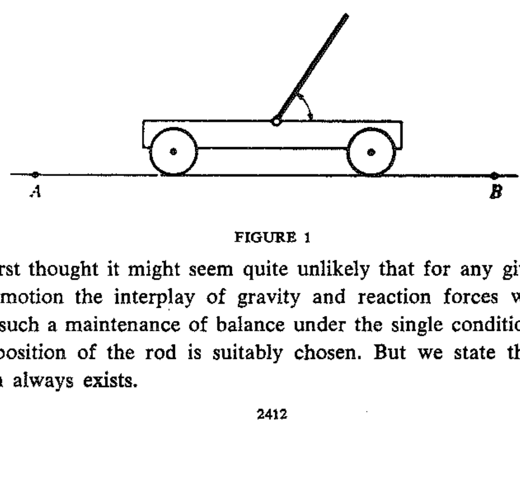


FIGURE 1

At first thought it might seem quite unlikely that for any given schedule of motion the interplay of gravity and reaction forces will always permit such a maintenance of balance under the single condition that the initial position of the rod is suitably chosen. But we state that such a position always exists.

Fortunately, the proof does not depend on a detailed knowledge of the laws of dynamics. (If it did, our task would be exceedingly difficult.) Only the following simple assumption of a physical nature need be granted: *The subsequent motion of the rod depends continuously on its initial position*; in particular, if for a given initial position the rod will fall down and hit the floor in one direction, then for any initial position differing sufficiently little from this, the rod will not hit the floor in the opposite direction.

Now the position of the rod in the train at any time is characterized by the angle α which it makes with the floor. To the angles $\alpha = 0°$ and $\alpha = 180°$ respectively correspond the two flat positions of the rod. We denote by x the angle of the initial position of the rod. The proof of our statement will be given indirectly, in line with its purely existential character. We shall assume that whatever initial position x we choose, the rod will *always* fall down and touch the floor, either at $\alpha = 0°$ or at $\alpha = 180°$. We may then define a function $f(x)$ whose value is to be $+1$ if the rod hits the floor at $\alpha = 0°$, and -1 otherwise. Since we have assumed that for each initial angle x we have one of the two cases, the function $f(x)$ is defined in the whole interval $0 \leqq x \leqq 180$. Obviously $f(0) = +1$ and $f(180) = -1$, while according to the assumed continuity property of our dynamical system, $f(x)$ will be a continuous function of x in the closed interval $0 \leqq x \leqq 180$. Hence, by Bolzano's theorem, it must have the value $f(x) = 0$ for some intermediate value of the initial angle x, contradicting the definition of $f(x)$ as only able to assume the values $+1$ or -1. This absurdity proves false the assumption that the rod will fall to the floor during the journey for every initial position x.

It is clear that this statement has an entirely theoretical character, since the proof gives no indication of how to find the desired initial position. Moreover, even if such a position could be calculated theoretically, it would probably be quite useless in practice, because of its instability. For example, in the extreme case where the train remains motionless at station A during the entire journey, the solution is obviously $x = 90°$, but anyone who has attempted to balance a needle upright on a plate for any length of time will find this result of little assistance. Nevertheless, for the mathematician, the existence proof that we have given loses none of its interest.