Study guide for the 3rd term test.

Coverage:

- All the material that was covered in the first and second term tests
- Inverse functions: Spivak's Chapter 12
- Lengths of curves: handout "Length of a curve"
  (Also see Spivak's Appendix to Chapter 12)
- Darboux integral: Spivak's Chapter 13
- Riemann integral: handout "The Riemann Integral"; Spivak's Appendix to Chap. 13
- Lengths of C^1 curves: handout "Length of a continuously differentiable curve".
  (Also see Spivak Appendix to Chapter 13, problems 2, 3, 4)
- The fundamental theorem of calculus: Spivak's Chapter 14
- Improper integrals: class notes.
  (Also see Spivak Chapter 14, problems 25, 26, 27, 28, 29, 30)
- Logarithm and exponential functions: Spivak's Chapter 18; handout "The logarithm and exponential functions"

If you will need to use the axioms for the real numbers, then I will provide them.

Sources.

- Your notes from class
- Your notes from tutorial
- Yael's notes from class
- Spivak's textbook
- Handouts on course website
- Weightless assignments
- For-credit assignments

Tentative list of relevant terms.

- Inverse functions; inverse function theorem
- Continuous monotone functions on intervals
- Partition of an interval; refinement of a partition; mesh of a partition
- Parametrized curve; rectifiable curve; length of a curve
- Properties of length: continuity (as a function of endpoints), concatenation, reparametrization, composition with rotations
- Lower and upper Darboux sums; lower and upper Darboux integrals, Darboux integrable function; Darboux integral of a function
- Properties of Darboux integrals: concatenation, comparison, linearity, continuity (as a function of endpoints)
- Tagging of a partition; tagged partition; Riemann sum; Riemann integrable function; Riemann integral of a function
- C^1 (continuously differentiable) function, C^1 (continuously differentiable) curve, velocity, speed, length as an integral of speed
- First and second fundamental theorems of calculus; applications
- Improper integrals on half-closed intervals and on open intervals (allowing infinite intervals)
- Logarithm and exponential functions: definitions and properties

Some practice questions from *earlier* material:

(1) Prove: if $\forall \epsilon > 0 \ \ |x - y| < \epsilon$, then $x = y$.

(2) Prove: if $\forall n \in \mathbb{N} \ \ |x - y| < \dfrac{1}{n}$, then $x = y$.

(3) For each of the following properties, determine which functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfy the property.

- $\forall x \ \exists y$ such that $f(x) > y$.
- $\exists y$ such that $\forall x \ f(x) > y$.
- $\exists x$ such that $\forall y \ f(x) > y$.
- $\forall y \ \exists x$ such that $f(x) > y$.

(4) Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be bounded from above. Let $a = \sup f$ and $b = \sup g$. Define $h(x) = \begin{cases} f(x) & \text{if } f(x) \geq g(x) \\ g(x) & \text{if } f(x) < g(x) \end{cases}$. (That is, $h = \max\{f, g\}$.) Prove that the function $h\colon \mathbb{R} \to \mathbb{R}$ is bounded from above and that $\sup h = \max\{a, b\}$.

(5) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bounded function. Prove that the limits $\displaystyle\lim_{x\to\infty}\left(\sup_{(x,\infty)} f\right)$ and $\displaystyle\lim_{x\to\infty}\left(\inf_{(x,\infty)} f\right)$ exist.

(These limits are called the "lim-sup" and "lim-inf" of $f(x)$ as $x \to \infty$, and they are often denoted $\displaystyle\overline{\lim_{x\to\infty}} f(x)$ and $\displaystyle\underline{\lim_{x\to\infty}} f(x)$. What are they when $f(x) = e^{-|x|}\sin x$? when $f(x) = e^{-1/|x|}\sin x$?)