MAT157Y TERM TEST 3 + COMMENTS

(1) Let $f\colon [3,7) \to \mathbb{R}$ be a continuous function. Let $A = \{x \in [3,7) \mid$ there exists $\delta > 0$ such that, for all $y, z \in [3,x]$, if $|y - z| < \delta$ then $|f(y) - f(z)| < 0.001\ \}$.

(a) Prove that $\sup A$ exists.

(b) What can you say about $\sup A$? Explain in 2–3 sentences.

Solution to (a): The set A is nonempty (it contains 3; make sure that you see why) and bounded from above (by 7). By the least upper bound axiom, A has a supremum.

Solution to (b): For each $x \in [3,7)$, the function f is uniformly continuous on $[3,x]$ (by a theorem), which implies that $x \in A$. So $A = [3,7)$. So $\sup A = 7$. (Make sure that you understand the vocabulary and can state the theorem.) (Partial solution: because $3 \in A$, $\sup A \geq 3$; because 7 is an upper bound for A, $\sup A \leq 7$.)

(2) Let $h\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function that satisfies $h'(x) = 1 + \cos^2(\sin x)$ for all x and $h(\pi) = 100$. (a) In 1–3 sentences, explain why h is one-to-one. (b) Let $g = h^{-1}$. Find $g'(100)$.

Solution to (a): $h' > 0$, so h is strictly monotone increasing, so h is one-to-one.

Main step of of solution to (b): $g'(100) = \dfrac{1}{h'(\pi)}$.

(3) Let $f(x) = \begin{cases} 10 & \text{if } 0 \leq x < 2 \\ -5 & \text{if } x = 2 \end{cases}$. Find a partition P of the interval $[0,2]$ and a number I such that the lower and upper Darboux sums satisfy $I - \frac{1}{50} < L(f,P) \leq U(f,P) < I + \frac{1}{50}$.

Idea: $I = 20$ and $P = \{0, t_1, 2\}$ with t_1 sufficiently close to 2.

(4) Express the area of a disc of radius 1 as an integral. Compute this integral.

Main steps of of solution: $2\int_{-1}^{1} \sqrt{1-x^2}dx$. Substitute $x = \sin u$.

(5) For each of the following statements, determine if it is true or false, and explain in one or two sentences.

(a) For any function $f\colon [0,1] \to \mathbb{R}$, if f is Darboux integrable and $\int_0^1 f(x)dx = 100$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n}) < 101$.

Solution: True. This follows from the definition of the Riemann integral, noting that the sum is a Riemann sum for a partition with mesh $= \frac{1}{n}$, and since Darboux integrability implies Riemann integrability with the same integral.

(b) For any function $f\colon \mathbb{R} \to \mathbb{R}$ that is strictly monotone increasing, there exists a function $g\colon \mathbb{R} \to \mathbb{R}$ such that $f(g(x)) = x$ for all $x \in \mathbb{R}$.

Solution: False. E.G., for $f(x) = e^x$, $f(g(x)) = x$ cannot hold for $x < 0$.

(c) For any function $f\colon [0,1] \to \mathbb{R}$, if f is Darboux integrable, then f is continuous.

Solution: False. E.G., $f(x) = \lfloor x \rfloor$.

(d) For any function $f\colon [0,1] \to \mathbb{R}$, if f is Darboux integrable, then $x \mapsto \int_0^x f(t)dt$ is continuous.

Solution: True. Since f is bounded, $x \mapsto \int_0^x f(t)dt$ is Lipschitz,. (We can't use here the fundamental theorem of calculus because f might not be continuous.)