MAT157Y TERM TEST 3, FEBRUARY 2019

Please answer all the questions; the four questions have equal weight. Clarity counts. Simplify your solutions when possible. A response "I don't know" and nothing more, to any question or section of a question, will give you 20% of the points for that question or section. You can answer the questions in any order, but please use separate pages for different questions, and mark the question number at the top corner of the page. You have 100 minutes. No aids allowed.

(1) For each of the following statements, determine if it is true or false, and add an explanatory sentence or diagram or formula. You do not need to give a complete proof.

(a) Let $f(x) = \begin{cases} 1 & \text{if } x \text{ is positive and irrational} \\ -1 & \text{if } x \text{ is negative and irrational} \\ 0 & \text{if } x \text{ is rational.} \end{cases}$

Then the Darboux integral $\int_{-1}^{1} f(x)dx$ exists and is equal to zero.

(b) The function $f(x) = \begin{cases} \dfrac{1-\cos x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ is continuous at $x = 0$.

(c) Define $F\colon [0,8] \to \mathbb{R}$ by $F(x) = \int_0^x f$ where $f\colon [0,8] \to \mathbb{R}$ is an integrable function. If f is differentiable at the point 4, then F is differentiable at the point 4.

(d) The improper integral $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$ converges.

(e) $\int_0^{\pi/2} \exp(\cos^2 x)dx = \int_{\pi/2}^{\pi} \exp(\sin^2 x)dx$.

(2) Compute the following anti-derivatives. Do not give complete proofs; do show your work.

(a) $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ (b) $\displaystyle\int \frac{1}{1-t^2}\,dt$ (c) $\displaystyle\int \cos^2 x\,dx.$

(3) Recall that a continuous curve $\gamma\colon [a,b] \to \mathbb{R}^2$ is **rectifiable** if the set

$$\mathcal{L} = \{\ell(\gamma, P) \mid P \text{ a partition of } [a,b]\}$$

is bounded from above, where $\ell(\gamma, P) = \sum_{j=1}^{n} \|\gamma(t_{j-1}) - \gamma(t_j)\|$ when $P = \{t_0, \ldots, t_n\}$; and that, if γ is rectifiable, then $\mathbf{length}(\gamma)$ is the supremum of the set $\mathcal{L}$.

Let $\gamma\colon [-2,2] \to \mathbb{R}^2$ be a rectifiable continuous curve. Let $\gamma_1\colon [0,1] \to \mathbb{R}^2$ be its restriction to the sub-interval $[0,1]$. Prove that γ_1 is also rectifiable, and that $\text{length}(\gamma_1) \leq \text{length}(\gamma)$.

(4) Let $f(x) = \lfloor x \rfloor$ =the maximal integer $\leq x$. Find a partition P of $[0,2]$ and a number I such that the lower and upper Darboux sums of f satisfy

$$I - \frac{1}{20} < L(f,P) \leq U(f,P) < I + \frac{1}{20}.$$

Show your work.