MAT157Y term test 3, February 2021

If anything happened or will happen today that might be interpreted as a suspected academic offence, by yourself or by another student, please ask for an individual appointment with Yael, to discuss what happened and whether and how to proceed.

Please submit on Crowdmark your handwritten copy of the declaration that is labeled as Question 0.

Please submit on Crowdmark your answers to Questions 1–4. These questions have equal weight.

A response "I don't know" and nothing more, to any question or section of a question, will give you 20% of the points for that question or section.

Clarity counts. Simplify your solutions when possible.

You have 100 consecutive minutes to write your solutions. After 100 minutes, stop writing. You will then have 10 additional minutes to upload your solutions on Crowdmark.

No aids allowed. (Nothing! No physical nor electronic note(s) or book(s); no access to any online materials; no calculator; no verbal nor written communication with anyone other than Yael Karshon or David Miyamoto.)

During the test, if you have a question, please send an email to Yael, `yael.karshon@utoronto.ca`, and to David, `david.miyamoto@mail.utoronto.ca`.

(0) Please copy the following declaration in handwriting, and sign your name below it:

> I confirm that I wrote this test, or will write this test, entirely on my own, without using any aids or assistance. I have not, and will not, communicate with anyone about the content of this test between 9am and 9pm EST today, other than the instructor or the TA.

(1) For each of the following statements, determine if it is true or false. Explain with a brief sentence or diagram or formula. You do not need to give a complete proof.

(a) The following function is Riemann integrable on the interval $[-1, 1]$:

$$f(x) := \begin{cases} n & \text{if } |x| = \frac{1}{n}, \text{ for } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

(b) The curve $\gamma(t) = (\sqrt{t}, t)$, for $t \in [0, 10]$, is rectifiable.

(c) For a function $f\colon [0, \infty) \to \mathbb{R}$, if the improper integral $\int_0^\infty f(x)dx$ converges, then f is bounded.

(d) The following function is differentiable at $x = 0$:

$$f(x) := \begin{cases} 0 & x \geq 0 \\ \frac{1}{x}e^{1/x} & x < 0 \end{cases}$$

(2) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function that satisfies $f'(x) = \frac{1}{\sqrt{1+x^4}}$.

(a) Why is f one-to-one?

Let $g := f^{-1}$ be the inverse function.

(b) Show that the function g satisfies $(g'(x))^2 - g(x)^4 = 1$.

(c) What can you say about the image (range) of g?

(d) What can you say about the domain of g?

Justify briefly.

(3) Let $f\colon [0, 8] \to \mathbb{R}$ be a bounded function. Suppose that f is Darboux integrable. Define $g\colon [0, 8] \to \mathbb{R}$ by $g(x) := \min\{f(x), 10\}$.

(a) Prove that g is Darboux integrable

(b) Prove that $\int_0^8 g(x)dx \leq 80$

(4) (a) Let $F(x) := \int_0^{x^2} \frac{dt}{1+\cos^5 t}$. Find an expression for F' without evaluating the integral.

(b) For $-1 \leq x \leq 1$, let $g(x) := \int_0^x f(t)dt$, where

$$f(x) := \begin{cases} \frac{1}{n} & \text{if } \frac{1}{n} < x \leq \frac{1}{n-1} \text{ for some integer } n \geq 2, \\ 0 & \text{otherwise .} \end{cases}$$

At which points x is $g'(x) = f(x)$? Justify briefly.