MAT157Y term test 4, April 2021

Please begin your response to each question on a new page.

Please submit on Crowdmark your handwritten copy of the declaration that is labeled as Question 0.

Please submit on Crowdmark your answers to Questions 1–4. These questions have equal weight.

A response "I don't know" and nothing more, to any question or section of a question, will give you 20% of the points for that question or section.

Clarity counts. Simplify your solutions when possible.

You have 100 consecutive minutes to write your solutions. After 100 minutes, stop writing. You will then have 10 additional minutes to upload your solutions on Crowdmark.

No aids allowed. (Nothing! No physical nor electronic note(s) or book(s); no access to any online materials; no calculator; no verbal nor written communication with anyone other than Yael Karshon or David Miyamoto.)

During the test, if you have a question, please send an email to Yael, `yael.karshon@utoronto.ca`, and to David, `david.miyamoto@mail.utoronto.ca`.

If anything happened or will happen today that might be interpreted as a suspected academic offence, by yourself or by another student, please ask for an individual appointment with Yael, to discuss what happened and whether and how to proceed.

(0) Please copy the following declaration in handwriting, and sign your name below it:

> I confirm that I wrote this test, or will write this test, entirely on my own, without using any aids or assistance. I have not, and will not, communicate with anyone about the content of this test between 9am and 9pm EST today, other than the instructor or the TA.

(1) (a) Let $h(x)$ be a function such that $h''(x) = -\frac{1}{2}h(x)$. Express the following indefinite integral in terms of the function h and/or its derivatives. Your expression should not involve integrals. Show your work.

$$\int e^x h(x)dx$$

(b) Find the following indefinite integral. Show your work.

$$\int \sin^{300}(x)\cos^5(x)dx.$$

(2) (a) Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers that converges to 5. Let $(n_k)_{k=1}^{\infty}$ be a sequence of natural numbers that diverges to ∞. For each k, let $b_k = a_{n_k}$. Using only the definitions of "converges to 5" and "diverges to ∞", prove that the sequence $(b_k)_{k=1}^{\infty}$ converges to 5.

(b) Let $f_n\colon [-1,1] \to \mathbb{R}$, for $n \in \mathbb{N}$, be a sequence of functions that converges uniformly to the zero function. Define $g_n\colon [0,2\pi] \to \mathbb{R}$ by $g_n(x) := f_n(\cos(x))$.
What can you say about the convergence of (g_n)? Justify.

(3) Your friend says this: "Because $\sin^2 x + \cos^2 x = 1$, we have that $\cos x = \sqrt{1-\sin^2 x}$ for all x. But the square-root function is not differentiable at the point 0, so the function $\cos x$ is not differentiable at the points where $\sin x = 1$."
Respond to your friend, in no more than a page.

(4) For each of the following statements, determine if it is true or false, and add an explanatory sentence or diagram or formula. You do not need to give a complete proof.

(a) "If $a_n \xrightarrow[n\to\infty]{} 2\pi$, then $\cos(a_n) \xrightarrow[n\to\infty]{} 1$"

(b) "The body of revolution that is obtained by rotating the disc

$$\{(x,y) \mid |x-100|^2 + |y-101|^2 \le 1\}$$

about the horizontal axis has the same volume as the body of revolution that is obtained by rotating this disc about the vertical axis"

(c) "If (a_n) is a sequence of non-negative numbers, and if the series $\sum_{k=1}^{\infty} a_{2k}$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges.

(d) "If (a_n) is a decreasing sequence of positive numbers, and if the series $\sum_{n=1}^{\infty} a_n$ diverges, then the series $\sum_{k=1}^{\infty} a_{2k}$ also diverges.

(e) "The series $\sum \frac{1+n^2}{n!}$ converges"