This weightless assignment is due on Crowdmark by Monday, October 26, at 9:00pm EST. It does not count toward your course grade.

Exercise 1. Read Spivak Chapter 4, Appendix 1, "Vectors." Consider the vectors $e := (0, 1)$ and $f := (1, 1)$.

(a) Given a vector $v = (v_1, v_2)$, can you find $a, b \in \mathbb{R}$ such that $v = ae + bf$?

(b) Suppose $a, b \in \mathbb{R}$ and $ae + bf = (0, 0)$. Is it possible that both $a \neq 0$ and $b \neq 0$?

If you take a course in linear algebra, you will recognize the set $\{e, f\}$ as a *basis* of the plane, when viewed as a *vector space*.

Exercise 2. Read Spivak Chapter 7, "Three Hard Theorems." Fill in the proof of the following theorem.

Theorem. Suppose $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $n > 0$ is even and $a_0 = f(0) < 0$. Then there are at least two numbers, x_1 and x_2, such that $f(x_1) = 0$ and $f(x_2) = 0$. In other words, f has at least two zeroes.

Proof. Choose

$$M := \max(\underline{\qquad}).$$

Then if $|x| > M$, we have $|x^k| > |x|$ and also

$$\frac{|a_{n-k}|}{|x^k|} < \frac{|a_{n-k}|}{|x|} < \underline{\qquad} = \frac{1}{2n}.$$

Consequently, by the $\underline{\qquad}$ and the inequality above,

$$\left|\frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right| \leq \underline{\qquad} < \frac{1}{2n} + \cdots \frac{1}{2n} = \frac{1}{2}.$$

This implies that, for $|x| > M$,

$$0 < \frac{x^n}{2} \leq x^n \left(1 + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a_0}{x^n}\right) = f(x).$$

In particular, if we choose $a := \underline{\qquad}$ and $b := \underline{\qquad}$, then $a > 0$ and $f(a) > 0$, and $b < 0$ and $f(b) > 0$. Because $f(0) < 0$, applying Theorem $\underline{\qquad}$ to the interval $\underline{\qquad}$ lets us conclude there is some $x_1 > 0$ such that $f(x) = 0$. Similarly, applying Theorem $\underline{\qquad}$ to the interval $\underline{\qquad}$ lets us conclude there is some $x_2 < 0$ such that $f(x) = 0$. As $x_2 < 0 < x_1$, we have found two distinct zeroes of f, as desired. $\square$