This weightless assignment is due on Crowdmark by Monday, November 9, at 9:00pm EST. It does not count toward your course grade.

Exercise 1. Read Spivak Chapter 8, "Least Upper Bounds."

(a) In the proof of Theorem 7-1 (the Intermediate Value Theorem), Spivak uses:

Lemma. Let $A \subseteq \mathbb{R}$ be a non-empty set, bounded from above by α. Then $\alpha = \sup A$ if and only if for every $\epsilon > 0$, there is some $x \in A$ such that $\alpha - \epsilon < x \leq \alpha$.

Prove this lemma, and indicate in which paragraph Spivak uses it.

(b) Suppose f is continuous from the right at a. Prove there is a $\delta > 0$ such that f is bounded on the set $[a, a+\delta)$. Must f be bounded on some set of the form $(a-\delta', a+\delta')$?

Exercise 2. Read Spivak Chapter 9, "Derivatives."

(a) Let $f(x) := x^2 \sin \frac{1}{x}$, where $f(0) := 0$. Consider the following argument that $f'(0)$ does not exist:

Proof.

$$
\begin{aligned}
f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} && \text{by definition of } f'(0) \\
&= \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} && \text{by definition of } f \\
&= \left(\lim_{h \to 0} h\right)\left(\lim_{h \to 0} \sin \frac{1}{h}\right) && \text{by the product rule for limits,}
\end{aligned}
$$

which does not exist because $\lim_{h \to 0} \sin \frac{1}{h}$ does not exist. □

Is this argument correct? If not, point out the error, and say whether $f'(0)$ exists or not.

(b) For each item below, give a function $f : [-1, 1] \to \mathbb{R}$ (with an expression or a clear drawing) with that property:

- f is not differentiable at any point (note we do not require continuity, so keep it simple).
- f is continuous, and both the left and right-hand derivative of f at 0 diverges to ∞.
- The left-hand derivative of f does not exist at 0 (and does not diverge to $\pm\infty$), and the derivative of f at $\frac{1}{2}$ is 1.

Exercise 3 (Complete Exercises 1 and 2 first)**.** Read Spivak Chapter 8, Appendix, "Uniform Continuity."

(a) Define what it means for f to be uniformly continuous on an interval A.

(b) In class, we defined what it means for f to be *Lipschitz* on A. Give the definition here.

(c) Suppose f is a function on $[a, b]$. Connect the concepts below with "implies" or "does not imply" arrows.

continuous uniformly continuous

Lipschitz