This weightless assignment is due on Crowdmark by Monday, March 22, at 9:00pm EST. It does not count toward your course grade.

Exercise 1. Read Spivak Chapter 24, "Uniform Convergence and Power Series."

(a) Suppose (f_n) is a sequence of functions on $[0, 1]$ such that $\lim_{n\to\infty} f_n(x)$ exists for each x; denote each limit by $f(x)$. Justify whether each statement below is true or false.

(i) If there is a sequence of points (x_n) such that $\lim_{n\to\infty} f_n(x_n)$ diverges to infinity, then there must be some x such that $f(x) > 0$ (i.e. the limit cannot be the zero function).

(ii) If each f_n is C^1, and (f_n) converges to f, and (f_n') converges uniformly to g, then f is differentiable and $f'(x) = g(x)$.

(iii) If each f_n is continuous, and (f_n) converges to f uniformly, then f is uniformly continuous.

(b) Give a sequence (f_n) of non-constant differentiable functions on $\mathbb{R}$ that satisfy the hypotheses of the Weierstrass M-Test. (Hint: you could try to find a smooth analog to Spivak's example). Explain why your choice works.

(c) Fix sequences (a_n) and (b_n).

(i) Suppose $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = l$, for some l. Why do $\sum_{n=1}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} b_n x^n$ converge on every closed subinterval of $(-1, 1)$? Must they necessarily converge on $[-1, 1]$?

(ii) Further assume that $\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} b_n x^n =: f(x)$. Prove that $a_k = b_k$ for each k.

(d) [Extra: Consider the functions $f_n(x) = \frac{1}{2^n}\{2^n x\}$ on $[0, 1]$, where $\{x\}$ denotes the distance from x to the nearest integer, as in Spivak. The functions converge uniformly to the zero function, whose graph has arclength 1. What is the arclength of each graph of f_n (say parametrized by $\gamma(t) := (t, f_n(t))$)? What does this mean for the relation between uniform convergence and arclength?]