This weightless assignment is due on Crowdmark by Monday, April 5, at 9:00pm EST. It does not count toward your course grade.

Exercise 1. Read Spivak Chapter 20, "Approximation by Polynomial Functions." Suppose f is at least twice differentiable on $\mathbb{R}$, and

$$\begin{aligned} f'' - f &= 0 \\ f(0) &= 0 \\ f'(0) &= 0. \end{aligned}$$

We will prove $f = 0$, in a similar way to Spivak's arguments about solving $f'' + f = 0$ near the end of the chapter (find them!).

Proof. The set $\{f^{(k)} \mid k \in \mathbb{N}\}$ has ____ elements [Hint: $f^{(3)} = (f'')' = f'$, by assumption on f. What about $f^{(4)}$?] Moreover,

$$f^{(k)} = \begin{cases} \underline{\qquad} & \text{if } k \text{ is } \underline{\qquad} \\ \underline{\qquad} & \text{if } k \text{ is } \underline{\qquad}. \end{cases}$$

In particular, $f^{(k)}(0) =$ ____ for all k. Fix n. This means $P_{n,0}(x) =$ ____. On the other hand, by Taylor's theorem, assuming for the moment $x > 0$,

$$R_{n,0}(x) = \underline{\qquad}, \text{ for some } t \in [0, x].$$

[Spivak writes unclearly here; he has a when it should be 0.] Since $f = P_{n,0} + R_{n,0}$, we conclude $f =$ ____.

Now, f and f' are continuous, so by the boundedness theorem on $[0, x]$, there exists some M_0 and M_1 such that

$$|f(t)| \leq M_0, \quad |f'(t)| \leq M_1, \quad \text{for all } t \in [0, x].$$

Therefore, $|f^{(n+1)}(t)| \leq$ ____ for all $t \in [0, x]$ [Hint: it should probably be bigger than both M_0 and M_1.] In particular, we can bound

$$R_{n,0}(x) \leq \underline{\qquad}.$$

For any $\varepsilon > 0$, we may therefore choose n so that $R_{n,0}(x) =$ ____ $< \varepsilon$. In other words, $|f(x)| < \varepsilon$ for all $\varepsilon > 0$, which means $|f(x)| = 0$. Since x was arbitrary and positive, we conclude $f = 0$ on $[0, \infty)$. A similar argument holds on $(-\infty, 0]$.

[Aside: this implies that given a, b, there is a unique function f satisfying $f'' - f = 0$ and $f(0) = a$ and $f'(0) = b$. Choosing $a = 0$ and $b = 1$, the unique solution f is called the "hyperbolic sine function," denoted sinh. Choosing $a = 1$ and $b = 0$, the unique solution f is called the "hyperbolic cosine function," denoted cosh.] □