The Poincaré upper half plane model for hyperbolic geometry[1]

The Poincaré upper half plane is an interpretation of the primitive terms of Neutral Geometry, with which all the axioms of Neutral geometry are true, and in which the hyperbolic parallel postulate is true. In this handout we will give this interpretation and verify most of its properties. We won't verify here all the axioms, but we'll verify enough of them to give you a reasonable grasp on this model.

If you find a mistake in this handout, please tell it to me so that I will fix it. I will give you extra bonus homework points for this.

We follow the theory of Neutral Geometry that is developed in John Lee's textbook *Aximatic Geometry*, 2013. It relies on the theory of sets and the theory of real numbers, as summarized in Appendices G and H of this book. The primitive terms for Neutral Geometry are **point**, **line**, **distance** between points, and **measure** of an angle. The postulates are summarized in Appendix D of the book.

With the Poincaré upper half-plane interpretation of the primitive terms "point", "line", "distance", and "angle measure", the postulates of Neutral Geometry become statements about real numbers. To show that this interpretation really is a model, we need to prove these statements about real numbers. Thus, the proofs in this document are proofs within the theory of real numbers (and *not* within the axiomatic theory of Neutral Geometry).

To avoid ambiguity, it will be convenient to use slightly different terminology for objects in the axiomatic theory and for their interpretation. We will use the terms h-point, h-line, h-distance, and h-angle measure for the Poincaré's upper half-plane interpretations of "point", "line", "distance", and "angle measure". We will often use the prefix h- (which stands for "hyperbolic") also for interpretations of other (non-primitive) terms.

"Points" and "lines" of the Poincaré upper half plane

The h-points and h-lines are described in Example 2.17 of our textbook, which we now recall.

An *h-point* is a pair (x, y) of real numbers such that $y > 0$. Thus, the set of h-points is the upper half-plane

$$\mathcal{H} \;=\; \{(x,y) \in \mathbb{R}^2 \mid y > 0\}.$$

An *h-line* is the intersection of the upper half-plane either with a vertical line or with a circle whose centre is on the x-axis. Thus, an h-line is either a set of the form

$$L_m \;:=\; \{(x,y) \in \mathcal{H} \mid x = m\}$$

where $m \in \mathbb{R}$, or a set of the form

$$L_{c,r} \;:=\; \{(x,y) \in \mathcal{H} \mid (x-c)^2 + y^2 = r^2\}$$

where $c \in \mathbb{R}$ and $r > 0$.

The **Set Postulate**, with the Poincaré upper half plane interpretation, becomes

Every h-line is a set of points, and there is a set of all h-points.

[1]Yael Karshon, MAT402, Fall 2016.

This statement is true; it follows from the above definitions of $\mathcal{H}$, L_m, and $L_{c,r}$.

Because every h-point is in particular an element of $\mathbb{R}^2$, we can refer to the x-coordinate and y-coordinate of an h-point.

Exercise 1. Fix $c \in \mathbb{R}$ and $r > 0$. Consider the h-line $L_{c,r}$. Note that $(x, y) \in L_{c,r}$ if and only if y is positive and $(x - c)^2 + y^2 = r^2$.

- Show that every two distinct h-points on $L_{c,r}$ have distinct x-coordinates.
- Show that the map $(x, y) \mapsto x$ defines a bijection from $L_{c,r}$ to the open interval $(c - r, c + r)$.

The **Unique Line Postulate**, with the Poincaré upper half plane interpretation, becomes

> Given any two distinct h-points, there exists a unique h-line that contains both of them.

To prove this statement, we need to first rephrase it as a statement about real numbers, by spelling out the meanings of h-point and h-line. A tricky aspect of this is that there are two possible descriptions of h-lines. (In the Cartesian model $\mathbb{R}^2$ there are also two descriptions of lines: a line could be given by an equation of the form $x = m$ or by an equation of the form $y = ax + b$. See Chapter 6 of the textbook.) Here is one way to rephrase the above statement that we need to prove:

> Given any two distinct points A and B in the upper half plane $\mathcal{H}$, exactly one of the following two possibilities occurs.
>
> (1) There exists a unique real number m such that A and B both lie on L_m, and there is no pair (c, r) with $c \in \mathbb{R}$ and $r > 0$ such that A and B both lie on $L_{c,r}$.
>
> (2) There exists a unique pair (c, r), with $c \in \mathbb{R}$ and $r > 0$, such that A and B both lie on $L_{c,r}$, and there is no $m \in \mathbb{R}$ such that A and B both lie on L_m.

We now prove this statement. Let $A = (x_A, y_A)$ and $B = (x_B, y_B)$ be distinct h-points. That is, $y_A > 0$ and $y_B > 0$, and A and B are distinct.

Case 1: suppose that $x_A = x_B$.

Let $m = x_A$. Then A and B both lie on L_m, and, for any $m' \neq m$, A and B don't both lie on $L_{m'}$ (moreover, neither of them lies on $L_{m'}$). Also, in this case there is no pair (c, r) with $c \in \mathbb{R}$ and $r > 0$ such that A and B both lie on $L_{c,r}$, because distinct points on $L_{c,r}$ have distinct x-coordinates (by Exercise 1) whereas A and B have the same x-coordinate (by assumption).

Case 2: suppose that $x_A \neq x_B$.

Then the Euclidean(=Cartesian) segment from A to B is not vertical. (Can you see why?) Let ℓ be its (Euclidean) perpendicular bisector; then ℓ is not horizontal. So ℓ meets the x-axis exactly once. Let $(c, 0)$ be the point where the ℓ meets the x-axis. Because $(c, 0)$ is on the (Euclidean) perpendicular bisector ℓ of the Euclidean segment from A to B, its Euclidean distance from A is equal to its Euclidean distance from B. Denote this distance by r. Then A and B are on the Euclidean circle with centre $(c, 0)$ and radius r. So they are on $L_{c,r}$. It remains to show that A and B are not on any other h-line. First suppose that they are on $L_{c',r'}$. Then $(c', 0)$, being of the same Euclidean distance (namely, distance r') from A and B, must be on the perpendicular bisector ℓ to the Euclidean segment from A to

B. But ℓ meets the x-axis in only the one point $(c, 0)$. So $(c', 0) = (c, 0)$, and then $r' = r$. Finally, because $x_A \neq x_B$, there is no m such that A and B are on L_m.

The **Existence Postulate**, with the Poincaré upper half plane interpretation, becomes

> There exist three distinct h-points such that no h-line contains all of them.

The purpose of the following exercise is to verify that the Existence Postulate is true in the Poincaré upper half plane interpretation.

Exercise 2. Let $A = (7, 1)$, $B = (7, 2)$, and $C = (2, 1)$. Find an h-line that passes through A and B. Show that this h-line does not pass through C. Prove that there is no h-line that contains all of these three h-points. (Hint: uniqueness.)

The Poincaré upper half plane interpretation has the hyperbolic parallel property. This fact can be verified directly. But, using the "all or nothing theorem" of neutral geometry, we can also deduce this fact by showing that the axioms of Neutral geometry are true in the Poincaré upper half plane model and that the Euclidean Parallel Property is false in the Poincaré upper half plane model. The purpose of the following exercise is to verify that the Euclidean parallel postulate is false in the Poincaré upper half plane interpretation.

Exercise 3.

- Draw two different pictures that illustrate the hyperbolic parallel property in the Poincaré upper half plane model. You do not need to provide proofs.
- Let ℓ be the h-line that is given by the equation $x^2 + y^2 = 1$, and let $A = (3, 3)$. Show that the equation $x = 3$ defines an h-line through A that does not meet ℓ. Show that the equation $(x - 7)^2 + y^2 = 25$ defines another h-line through A that does not meet ℓ.

Endpoints of an h-line

Points on the x-axis are not h-points. But we can still refer to them as points in $\mathbb{R}^2$.

We think of the h-line $L_{c,r}$ as having the endpoints $(c - r, 0)$ and $(c + r, 0)$. (Please make sure that you see why.) These are not h-points. They are points on $\mathbb{R}^2$ but they are not in the upper half plane. We can think of them as "points at infinity" of $L_{c,r}$, but – warning –

- Since the Euclidean Parallel Property is false in the Poincaré upper half plane, we do not have "transitivity of parallelism", and we cannot make sense of the projective completion in the same way that we did for Euclidean planes earlier on in our course.
- In the projective completion of a Euclidean plane, each line had only one "point at infinity". In our current situation, the h-line $L_{c,r}$ has two distinct endpoints.

To consider endpoints of the h-line L_m, we introduce the *extended x-axis*,

$$(\text{the x-axis}) \sqcup \infty.$$

We think of the h-line L_m has having the endpoints $(m, 0)$ and ∞. (Please make sure that you see why.)

We denote the Euclidean distance in $\mathbb{R}^2$ by $d(\cdot, \cdot)$. Thus, for $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$, we have $d(P, Q) = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}$.

Exercise 4. Show that, for any two distinct elements A', B' of the extended x-axis, there exists a unique h-line with endpoints A' and B'. (Hint: consider two cases – one where A' or B' is ∞, and another where A' and B' are both on the x-axis.)

Cross ratio

Let A', A, B, B' be any four points in $\mathbb{R}^2$ such that $A \neq A'$, $A \neq B'$, $B \neq A'$, and $B \neq B'$. Define their **cross-ratio** to be

$$(A', A, B, B') = \frac{d(A', B)\, d(A, B')}{d(A', A)\, d(B, B')}.$$

We note that, with the usual identification of $\mathbb{R}^2$ with the set $\mathbb{C}$ of complex numbers, the upper half plane becomes the set of complex numbers with positive imaginary part, and the x-axis becomes the set $\mathbb{R}$ of real numbers viewed as a subset of the set $\mathbb{C}$ of complex numbers. In the literature there is the notion of the cross-ration of four complex numbers, which can be defined as $(z_1, z_2, z_3, z_4) = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_2)(z_3-z_4)}$. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, the cross ratio in our sense becomes the absolute value of the cross-value in the complex sense, because $\left|\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_2)(z_3-z_4)}\right| = \frac{|z_1-z_3||z_2-z_4|}{|z_1-z_2||z_3-z_4|}$.

We will be interested in the cross-ration (A', A, B, B') when A, B are h-points on an h-line and A', B' are the endpoints of this h-line (so A', B' themselves are not h-points). We will want to also allow A' or B' to be the element ∞ of the extended x-axis. When A' or B' is ∞, we define

$$(A', A, B, \infty) = \frac{d(A', B)}{d(A', A)} \quad \text{and} \quad (\infty, A, B, B') = \frac{d(A, B')}{d(B, B')}.$$

This is consistent with our previous formula for the cross-ratio if we take the informal conventions that for any h-point P we have $d(P, \infty) = \infty$ and that $\frac{\infty}{\infty} = 1$. For example, with these informal conventions, we can write

$$(A', A, B, \infty) = \frac{d(A', B)\, d(A, \infty)}{d(A', A)\, d(B, \infty)} = \frac{d(A', B)}{d(A', A)} \cdot \frac{\infty}{\infty} = \frac{d(A', B)}{d(A', A)}.$$

Exercise 5. Prove the following three properties of the cross-ratio in each of the three cases below.

(a) The cross-ration (A', A, B, B') is a positive real number.
(b) If $A = B$, then the cross-ration (A', A, B, B') is equal to 1.
(c) $(B', A, B, A') = \dfrac{1}{(A', A, B, B')}$.

The three cases are

(i) A', A, B, B' are points in $\mathbb{R}^2$ such that $A \neq A'$, $A \neq B'$, $B \neq A'$, and $B \neq B'$.
(ii) A', A, B are points in $\mathbb{R}^2$ such that $A \neq A'$ and $B \neq A'$, and $B' = \infty$.
(iii) A, B, B' are points in $\mathbb{R}^2$ such that $A \neq B'$ and $B \neq B'$, and $A' = \infty$.

"Distance" in the Poincaré upper half plane

Now, let A, B be any two h-points. If $A = B$, we define $\text{h-dist}(A, B) = 0$. If $A \neq B$, then A and B determine a unique h-line; let A' and B' be the endpoints of this h-line; we then define

$$\text{h-dist}(A, B) = |\ln(A', A, B, B')|$$

where $\ln(\cdot)$ is the natural logarithm function and (A', A, B, B') is the cross-ratio. Because (A', A, B, B') is positive, its natural logarithm is a well defined real number, and the absolute value $|\ln(A', A, B, B')|$ is a well-defined non-negative real number. But the association of the symbols A' and B' to the two endpoints of the h-line was arbitrary. For the definition of h-distance to be unambiguous, we need to confirm that the expression $|\ln(A', A, B, B')|$ that defines the h-distance does not change its value when we switch A' and B'. And, indeed, since

$$(B', A, B, A') = \frac{1}{(A', A, B, B')}$$

(by Exercise 5), we have $\ln(B', A, B, A') = -\ln(A', A, B, B')$, which implies that $|\ln(B', A, B, A')| = |\ln(A', A, B, B')|$, as required.

We have now shown that the Distance Postulate holds in the Poincaré upper half plane interpretation:

> For every pair of h-points A and B, the h-distance h-dist(A, B) is a non-negative real number determined by A and B.

Note that for any h-line and any two points A, B on this h-line, if A', B' are the two endpoints of the h-line, then

$$\text{h-dist}(A, B) = |\ln(A', A, B, B')|.$$

If $A \neq B$, this equality is true by definition. If $A = B$, the left hand side of this equality is zero by definition, and the right hand side is zero because $(A', A, B, B') = 1$ whenever $A = B$ (by Exercise 5).

Next, we would like to verify the Ruler Postulate in the Poincaré upper half plane interpretation:

> For every h-line L, there exists a function $f\colon L \to \mathbb{R}$ that is a bijection and such that for any two h-points A, B on L we have h-dist$(A, B) = |f(A) - f(B)|$.

Our verification of the ruler postulate will use the two facts that are listed in the following exercise.

Exercise 6. Explain each of the following two facts in one paragraph. You don't need to provide a formal proof; just give an explanation that a first year calculus student will find convincing.

(1) The formula $x \mapsto \dfrac{1+x}{1-x}$ defines a bijection from the interval $(-1, 1)$ to the set $(0, \infty)$ of positive real numbers.
(2) The formula $y \mapsto \ln y$ defines a bijection from the set of positive real numbers to the set of all real numbers.

Confirmation of the Ruler Postulate, part 1.

We begin with an h-line of the form L_m. Its endpoints are $A' = (m, 0)$ and $B' = \infty$. Let A, B be two h-points on L_m. Then $A = (m, y_A)$ and $B = (m, y_B)$ for some positive real numbers y_A and y_B. We have $(A', A, B, B') = \dfrac{d(A', B)}{d(A', A)} = \dfrac{y_B}{y_A}$. So

$$\text{h-dist}(A, B) = \left|\ln \frac{y_B}{y_A}\right|.$$

Define $f\colon L_m \to \mathbb{R}$ by

$$f(P) = \ln y_P$$

where y_P is the y-coordinate of P. We now show that this is a coordinate function.

- From the definition of L_m, the function $P \mapsto y_P$ is a bijection from L_m to the set of positive real numbers. As noted earlier, the function $y \mapsto \ln y$ is a bijection from the set of positive real numbers to the set of all real numbers. The function f, being the composition of these two bijections, is a bijection.
- For any A, B in L_m, writing $A = (m, y_A)$ and $B = (m, y_B)$, we have

$$\text{h-dist}(A, B) = \left|\ln \frac{y_B}{y_A}\right| = |\ln y_B - \ln y_A| = |f(B) - f(A)|,$$

as required.

Confirmation of the Ruler Postulate, part 2.

Next, we consider h-lines of the form $L_{c,r}$. We begin with the special case that $c = 0$ and $r = 1$, so that the h-line is the intersection of the upper half plane with the circle that is centred at the origin and has radius one.

The first step is to work out the formula for the h-distance. The endpoints are $A' = (-1, 0)$ and $B' = (1, 0)$. (Can you see why?) Let A, B be any two h-points on this h-line. Write them as $A = (x_A, y_A)$ and $B = (x_B, y_B)$. Note that y_A and y_B are positive, and $x_A^2 + y_A^2 = x_B^2 + y_B^2 = 1$. (Can you see why?) We calculate the Euclidean distances that appear in the formula for the cross-product.

$$\begin{aligned}
d(A', B) &= \sqrt{(1 + x_B)^2 + y_B^2} = \sqrt{2 + 2x_B} \\
d(A, B') &= \sqrt{(1 - x_A)^2 + y_A^2} = \sqrt{2 - 2x_A} \\
d(A', A) &= \sqrt{(1 + x_A)^2 + y_A^2} = \sqrt{2 + 2x_A} \\
d(B, B') &= \sqrt{(1 - x_B)^2 + y_B^2} = \sqrt{2 - 2x_B}.
\end{aligned}$$

Please make sure that you can justify each of the above equalities. We now substitute these in the formula for the cross-ratio:

$$(A', A, B, B') = \frac{d(A', B)d(A, B')}{d(A', A)d(B, B')} = \frac{\sqrt{2 + 2x_B}\,\sqrt{2 - 2x_A}}{\sqrt{2 + 2x_A}\,\sqrt{2 - 2x_B}} = \left(\frac{(1 + x_B)(1 - x_A)}{(1 + x_A)(1 - x_B)}\right)^{\frac{1}{2}}.$$

To obtain the h-distance, we take the absolute value of the natural logarithm:

$$\begin{aligned}
\text{h-dist}(A, B) = |\ln(A', A, B, B')| &= \left|\ln\left(\left(\frac{(1 + x_B)(1 - x_A)}{(1 + x_A)(1 - x_B)}\right)^{\frac{1}{2}}\right)\right| \\
&= \frac{1}{2}\left|\ln\left(\frac{1 + x_B}{1 - x_B}\right) - \ln\left(\frac{1 + x_A}{1 - x_A}\right)\right|.
\end{aligned}$$

Again, please make sure that you can justify these algebraic manipulations; if you're not sure then ask.

Still for $L = L_{c,r}$ with $c = 0$ and $r = 1$, define

$$f\colon L \to \mathbb{R}$$

by

$$f(P) = \frac{1}{2} \ln \frac{1 + x_P}{1 - x_P}$$

where x_P is the x-coordinate of P. We now show that this is a coordinate function.

- The function $P \mapsto x_P$ is a bijection from L to $(-1, 1)$, the function $x \mapsto \dfrac{1+x}{1-x}$ is a bijection from $(-1, 1)$ to the set of positive real numbers, and the natural logarithm is a bijection from the set of positive real numbers to the set of all real numbers (see Exercise 6). Because the function f is the composition of these three bijections, it is a bijection.
- For any A, B in L, denoting their x-coordinates by x_A and x_B, we have

  $$\text{h-dist}(A, B) = \frac{1}{2} \left| \ln \left(\frac{1 + x_B}{1 - x_B} \right) - \ln \left(\frac{1 + x_A}{1 - x_A} \right) \right| = |f(B) - f(A)|,$$

  as required.

Confirmation of the Ruler Postulate, part 3.

Next, we consider a general h-line of the form $L_{c,r}$.

The map

$$g(x, y) = (\frac{1}{r}(x - c), \frac{1}{r} y)$$

is a bijection from the h-line $L_{c,r}$ to the h-line $L_{0,1}$.

Indeed, subtracting c from the x-coordinate creates a shift of distance c to the left. So it takes the circle of radius r centered at $(c, 0)$ to the circle of radius r centred at the origin. And multiplying by $\frac{1}{r}$ takes the circle of radius r centred at the origin to the circle of radius 1 centred at the origin. (Can you see why?) Finally, both of these maps take the upper half plane onto itself. So their composition g takes $L_{c,r}$ to $L_{0,1}$. To show that it is a bijection, it is enough to find an inverse. And, indeed, the map $h(x, y) \mapsto (c+rx, ry)$ takes the h-line $L_{0,1}$ to the h-line $L_{c,r}$ and is an inverse to g: we have $(x', y') = g(x, y)$ if and only if $(x, y) = h(x', y')$.

Exercise 7. In this exercise you will show that the map g is an isometry, which means that it does not distort h-distance: for any two h-points A and B, h-dist$(A, B) =$ h-dist$(g(A), g(B))$.

(a) Show that, for every P and Q in $\mathbb{R}^2$, we have $d(g(P), g(Q)) = \frac{1}{r} d(P, Q)$.

(b) Conclude that, for any A', A, B, B' in $\mathbb{R}^2$ such that each of A', B' is different from A and from B, we have $(g(A'), g(A), g(B), g(B')) = (A', A, B, B')$.

(c) For any two distinct h-points A and B, take A' and B' to be the endpoints of the h-line through A and B, and use Part (b) to show that h-dist$(g(A), g(B)) =$ h-dist(A, B).

We obtain a coordinate function for $L_{c,r}$ by composing the map g with a coordinate function for $L_{0,1}$. Namely, we take a coordinate function $f_0 \colon L_{0,1} \to \mathbb{R}$ for the h-line $L_{0,1}$ (such as the one that we described earlier), and on the h-line $L_{c,r}$ we take the composition

$$f := f_0 \circ g \colon L_{c,r} \to \mathbb{R}.$$

We now show that this is a coordinate function.

- The function g is a bijection from $L_{c,r}$ to $L_{0,1}$, as we have shown, and the function f_0 is a bijection from $L_{0,1}$ to $\mathbb{R}$, because it's a coordinate function. Because f is a composition of these two bijections, it is a bijection.

- For any A, B in $L_{c,r}$, we have

  $$\text{h-dist}(A,B) = \text{h-dist}(g(A), g(B)) = |f_0(g(A)) - f_0(g(B))| = |f(A) - f(B)|.$$

  The first equality is by Exercise 7, the second is because f_0 is a coordinate function on $L_{0,r}$, and the third is from the definition of f. We conclude that f is a coordinate function on $L_{c,r}$.

Plane Separation

The two sides of an h-line L_m are the sets

$$\{(x,y) \in \mathcal{H} \mid x < m\} \quad \text{and} \quad \{(x,y) \in \mathcal{H} \mid x > m\}.$$

The two sides of an h-line $L_{c,r}$ are the sets

$$\{(x,y) \in \mathcal{H} \mid y > \sqrt{r^2 - (x-c)^2}\} \quad \text{and} \quad \{(x,y) \in \mathcal{H} \mid y < \sqrt{r^2 - (x-c)^2}\}.$$

The proof that these sets satisfy the properties that are required in the Plane Separation postulate uses the intermediate value theorem of calculus (applied to a parametrization of the h-segment between two h-points that are not on the h-line). We omit the details.

Rays in the Poincaré upper half plane

The h-rays that lie on the h-line L_m and start from the h-point (m, y_A) are

$$\{(x,y) \mid x = m \text{ and } y \geq y_A\}$$

and

$$\{(x,y) \mid x = m \text{ and } 0 < y \leq y_A\}.$$

The h-rays that lie on the h-line $L_{c,r}$ and start from the h-point (x_B, y_B) are

$$\{(x,y) \mid c - r < x \leq x_B \text{ and } y = \sqrt{r^2 - (x-c)^2}\}$$

and

$$\{(x,y) \mid x_B \leq x < c + r \text{ and } y = \sqrt{r^2 - (x-c)^2}\}.$$

In particular, an h-ray that starts from point A is a portion of a circle or a line in $\mathbb{R}^2$ that starts from point A.

Figure 1 gives illustrations of h-rays. Please make sure that you understand how these illustrations relate to the above formulas.

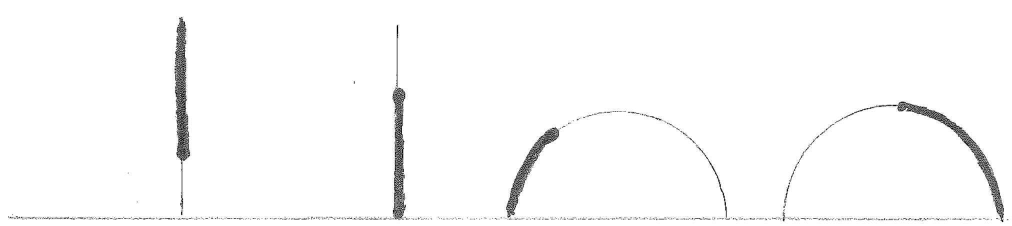

Figure 1. h-rays

Angle measures in the Poincaré upper half plane

Finally, we need to give the interpretation of angle measure. This is easy – the angle measure in the Poicaré upper half plane is the ordinary (Euclidean) angle measure between the corresponding portions of circles or lines.

We give more details.

Let γ be an h-line and let A be a point on γ. An h-ray that starts at A and lies on γ determines a Euclidean ray that starts at A and lies on the (Euclidean) tangent to γ at A.

For example, the h-ray $\{(x, y) \mid x = m \text{ and } y \geq y_A\}$ is itself a Euclidean ray, and the h-ray $\{(x, y) \mid x = m \text{ and } 0 < y \leq y_A\}$ determines the Euclidean ray $\{(x, y) \mid x = m \text{ and } y \leq y_A\}$.

The tangent line to $y = \sqrt{r^2 - (x - c)^2}$ at the point (x_A, y_A) is

$$\left\{(x, y) \mid y = y_A - \frac{x_A - c}{y_A}(x - x_A)\right\}.$$

(Check this!) The h-ray that lies on $L_{c,r}$ and starts at (x_A, y_A) and whose points have x-coordinates $\leq x_A$ determines the Euclidean ray that lies on the tangent line to L_{c-r} at (x_A, y_A) and whose points have x-coordinates $\leq x_A$. Similarly, the h-ray that lies on $L_{c,r}$ and starts at (x_A, y_A) and whose points have x-coordinates $\geq x_A$ determines the Euclidean ray that lies on the tangent line to L_{c-r} at (x_A, y_A) and whose points have x-coordinates $\geq x_A$.

An h-angle consists of two h-rays in $\mathcal{H}$ that start at the same point. Each of these determines a Euclidean ray in $\mathbb{R}^2$ that starts at that point, so together they determine a Euclidean angle in $\mathbb{R}^2$ with vertex at that point. We define the h-angle measure of the given h-angle to be the ordinary (Euclidean) angle measure of the angle that it determines.

With this interpretation, the Angle Measure postulate in the Poincaré upper half plane is true: for every h-angle, its h-angle measure is a real number in the closed interval $[0, 180]$ that is determined by the h-angle.

The Protractor postulate in the Poincaré upper half plane is a consequence of the protractor postulate in the Cartesian plane $\mathbb{R}^2$ (which is sketched in the textbook), together with the following fact:

> For every h-point, every Euclidean ray that starts from that point corresponds to some h-ray that starts from that point.

It is convenient to encode a Euclidean ray by its unit tangent vector that is based at the starting point of the ray and points in the direction of the ray. See Figures 2 and 3.

We then claim that for every point z in $\mathcal{H}$ and every direction v at z, there exists a unique h-ray emanating from z in the direction v.

If vector v is vertical, it's easy; see Figure 2.

Explicitly, if $z = (m, n)$ and $v = (0, 1)$, we take the h-ray $\{(m, y) \mid y \geq n\}$, and if $z = (m, n)$ and $v = (0, -1)$, we take the h-ray $\{(m, y) \mid 0 < y \leq n\}$.

If vector v is not vertical, we seek a Euclidean circle through z, centered at the origin, whose tangent at z is v. Because the tangent of a circle is perpendicular to its radius, we consider the Euclidean line through z in the direction perpendicular to v. This line meets the x-axis; let $(c, 0)$ be the point of intersection. Then v is tangent at z to the circle through z centered at $(c, 0)$. The intersection of this circle with the upper half plane $\mathcal{H}$ is the h-line $L_{c,r}$, where r is the Euclidean distance between z and $(c, 0)$. Part of this h-line forms an h-ray that emanates from z in the direction v.

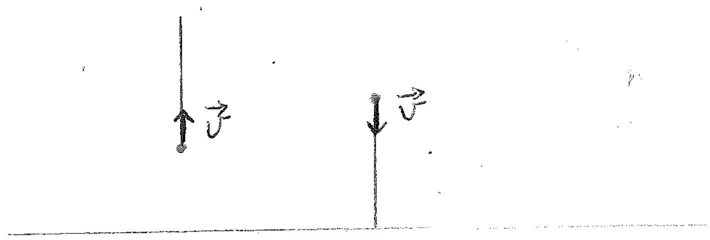

FIGURE 2. h-rays in vertical direction

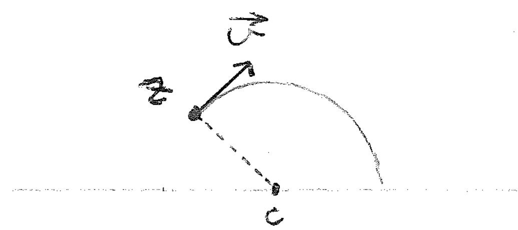

FIGURE 3. Seeking an h-ray in direction v at point z.

The SAS postulate is also true in the Poincaré upper half plane interpretation. To show it, we need to examine the *isometries* of the Poincaré upper half plane: the maps that preserve h-distance. These include the map g that we used to verify the ruler postulate, as well as so-called hyperbolic reflections and hyperbolic rotations. Once we obtain a good enough understanding of these maps, we can prove that the SAS postulate is true in the Poincaré upper half plane by imitating Euclid's idea of his "proof" of SAS. We do not have the space and time to discuss here the details.