MAT402 Classical Geometries, Fall 2016. Problem set 7

Please re-read all of Chapter 4 from John Lee's textbook. This problem set is due in class on Tuesday Nov.8st. You are encouraged to work in a group, but you must write your solution later, separately, on your own.

- If you worked in a group, please indicate with whom you worked.
- Copy the following sentence, and sign it when you're done preparing your submission: "I declare that I wrote these solutions entirely on my own."

(1) Trichotomy for betweenness of rays (textbook Corollary 4.9) is this:

> Let $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ be rays with a common endpoint. Suppose that no two of these rays are collinear and that there exists a half rotation that contains all of them. Then exactly one of the three rays lies between the other two.

("Exactly one" means at least one and at most one. Note: you're not asked to prove anything.) *Give a statement about real numbers* that implies that, in this setup, *at least* one of the three rays lies between the other two.

(2) Consistency of betweenness of rays (textbook Corollary 4.10) is this:

> Let $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ be distinct rays with a common endpoint. Suppose that no two of these rays are collinear. Suppose that there exists a half rotation that contains all of them; let $\mathrm{HR}(\overrightarrow{r}, P)$ be such a half rotation and let $g\colon \mathrm{HR}(\overrightarrow{r}, P) \to [0, 180]$ be a corresponding coordinate function. Then $\overrightarrow{a} \star \overrightarrow{b} \star \overrightarrow{c}$ if and only if $(g(\overrightarrow{a}) < g(\overrightarrow{b}) < g(\overrightarrow{c})$ or $g(\overrightarrow{a}) > g(\overrightarrow{b}) > g(\overrightarrow{c})$).

The direction "if" follows immediately from the definition of "between" for rays. *Prove the direction "only if"* according to the following guidelines. Assume $\overrightarrow{a} \star \overrightarrow{b} \star \overrightarrow{c}$. Let $x = g(\overrightarrow{a})$, $y = g(\overrightarrow{b})$, and $z = g(\overrightarrow{c})$. We need to prove that $x < y < z$ or $x > y > z$. (This is *not* a direct consequence of the assumption $\overrightarrow{a} \star \overrightarrow{b} \star \overrightarrow{c}$; make sure that you understand why.)

(a) Quote an earlier result to show that neither $\overrightarrow{b} \star \overrightarrow{a} \star \overrightarrow{c}$ nor $\overrightarrow{a} \star \overrightarrow{c} \star \overrightarrow{b}$ holds.
(b) Quote an earlier definition to deduce that none of the following inequalities hold: $y < x < z$, $\quad y > x > z$, $\quad x < z < y$, $\quad x > z > y$.
(c) Explain why $x \neq y \neq z \neq x$. (Make sure that you understand why is it not enough to just write $x \neq y \neq z$.)
(d) State a property of the real numbers that implies that $x < y < z$ or $x > y > z$.

(3) Prove Theorem 4.11 from the textbook – Euclid's common notions for angles. (Hint: you don't need to work with coordinate functions for half rotations. Just use properties of angle measures and the "betweenness theorem for rays".)

(4) Prove Theorem 4.29 from the textbook – the four right angles theorem. (Make sure that you understand why this is not obvious from the definition of "perpendicular".)

When proving a theorem from the book, you may rely on theorems that appear earlier in the book but not on theorems that appear later in the book.

Additional questions, for you to solve but not to hand in:

- Prove Theorem 4.17 from the textbook – the vertical angle theorem.
- Prove Theorem 4.30 from the textbook – construction of a perpendicular.
- Here is a flawed proof of the theorem "There exist three distinct lines" of incidence geometry. *Explain the flaw.*

  Let A, B, C be three distinct points; such points exist by Axiom I-1. Since A, B are distinct points, by Axioms I-2 and I-3 there is a unique line, $\overleftrightarrow{AB}$, that passes through them. Similarly, there is a unique line $\overleftrightarrow{AC}$ that passes through A, C and there is a unique line $\overleftrightarrow{BC}$ that passes through B, C. Because these lines are unique, they must be distinct.