MAT402 Classical Geometries, Fall 2016. Problem set 8

John Lee's textbook: read Chapter 5.

This problem set is due in class on Friday Nov.18th. You are encouraged to work in a group, but you must write your solution later, separately, on your own.

- If you worked in a group, please indicate with whom you worked.
- Copy the following sentence, and sign it when you're done preparing your submission: "I declare that I wrote these solutions entirely on my own."

As usual, when proving a theorem in Neutral geometry that appears in our textbook, you may use parts of Neutral geometry that appear earlier in the textbook. These may include postulates, theorems (or lemmas/propositions/corollaries etc), primitive notions, and definitions.

(1) Prove the "Consistency of triangle vertices" – Theorem 5.1 in the textbook: In a triangle $\triangle ABC$, the points A, B, and C are extremal and all the other points are not extremal.

(2) Solve Exercise 5C on Page 122 of the textbook. (It's about the intersections of a line with the sides of a triangle.)

(3) Prove the Converse to the Isosceles Triangle Theorem, which is Theorem 5.8 of the textbook. Do this by following the ideas of Euclid's proof of his Proposition I.6 while staying within the textbook formalism of Neutral Geometry.

(4) Prove the Triangle Inequality: if A, B, and C are non-collinear points, then $AC < AB + BC$. This is Theorem 5.18 in the textbook; see the hints that are provided there.

(5) Consider the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$. Define a *coordinate function* on S^1 to be a function $\theta\colon A \to [a, b]$ from a subset A of S^1 to an interval $[a, b]$ of real numbers such that θ is a bijection and such that the inverse $\theta^{-1}\colon [a, b] \to A$ is given by the formula $\theta^{-1}(t) = (\cos t, \sin t)$. For points p, q, r on S^1 define "q is between p and r", and write $p \star q \star r$, if there exists a coordinate function $\theta\colon A \to [a, b]$ such that $p, q, r \in A$ and such that $\theta(p) < \theta(q) < \theta(r)$ or $\theta(p) > \theta(q) > \theta(r)$.

  (i) Let $p = (1, 0)$, $q = (0, 1)$, and $r = (-1, 0)$. Show that $p \star q \star r$ and $p \star r \star q$ are both true.

  (ii) Find the flaw in the following flawed proof of the false theorem "$p \star q \star r$ and $p \star r \star q$ cannot both be true":
  Seeking a contradiction, suppose $p \star q \star r$ and $p \star r \star q$ both hold. Let A be a subset of S^1 that contains p, q, r, let $[a, b]$ be an interval of real numbers, and let $\theta\colon A \to [a, b]$ a coordinate function. Let $x = \theta(p)$, $y = \theta(q)$, $z = \theta(r)$. From $p \star q \star r$, we conclude that $x < y < z$ or $x > y > z$. From $p \star r \star q$, we conclude that $x < z < y$ or $x > z > y$. From properties of real numbers, ($x < y < z$ or $x > y > z$) and ($x < z < y$ or $x > z > y$) cannot be true. This gives our desired contradiction.