GEOMETRY OF POINTS, LINES, AND PLANES IN THREE DIMENSIONAL EUCLIDEAN SPACE

YAEL KARSHON, MAT402, FALL 2016

We work in the standard three dimensional Euclidean space, which we can identify with $\mathbb{R}^3$.
Points or sets of points in space are **collinear** if there is a line that contains all of them.
Points or sets of points in space are **coplanar** if there is a plane that contains all of them.
Two lines in space are **parallel** if they are coplanar and don't meet.[1]
A line and a plane in space are **parallel** if they don't meet.
Two planes in space are **parallel** if they don't meet.
Two lines in space are **skew** if they are not coplanar.
Portions of lines (such as segments or rays), or portions of planes (such as faces of a cube), are **parallel** if, when extended indefinitely, the resulting lines or planes are parallel.

Points

- Given two distinct points, there exists a unique line that contains both of them.
  For any plane, if the two points lie on the plane, then the line that contains them is entirely contained in the plane.
- Given three non-collinear points, there exists a unique plane that contains all of them.

Two lines

- Given two lines, exactly one of four possibilities occurs:
  - The lines coincide.
  - The lines meet at a unique point. Then, there exists a unique plane that contains both lines.
  - The lines are parallel. Then, there exists a unique plane that contains both lines.
  - The lines are skew.
- Given two parallel lines and a third line that is distinct from both, the third line is parallel to the first if and only if it is parallel to the second.

A line and a plane

- Given a line and a plane, exactly one of three possibilities occurs:
  - The line is contained in the plane.
  - The line meets the plane at a unique point.
  - The line is parallel to the plane.
- Given a plane, and given a line that is contained in the plane or is parallel to it, for each point in the given plane, the line through it that is parallel to the given line is entirely contained in the given plane.
- Given a plane and a line in it, every line that is parallel to the given line is either entirely contained in the given plane or is parallel to it.

[1] Note that we require parallel lines to be coplanar. This is consistent with Euclid's definition of "parallel". This requirement doesn't appear in the notion of "parallel" in the formal axiomatic theory of Incidence Geometry that we studied in class; in this context of Incidence Geometry we think of all points and lines as lying in one single plane.

Two planes

- Given two planes, exactly one of three possibilities occurs:
  - The planes coincide.
  - The planes meet in a line.
  - The planes are parallel.
- Given two parallel planes and a third plane that is distinct from both,
  - The third plane is parallel to the first if and only if it is parallel to the second.
  - If the third plane is not parallel to the first and second planes, the line where it intersects the first plane is parallel to the line where it intersects the second plane.

Two planes and a line

- Given two planes that intersect in a line,
  - If a line is parallel to the two planes, then it is also parallel to (in particular coplanar with) the intersection of the two planes.
  - If a line lies in the first plane and is parallel to the second plane, then it is also parallel to the intersection of the two planes.

"Sweeping" a plane by lines

- Given a plane and a point not on it, there exists a unique plane through the given point that is parallel to the given plane. This plane is equal to the union of the lines through the given point that are parallel to the given plane.

- Given a line and a point not on it, there exists a unique plane that contains the given line and the given point. This plane is equal to the union of the following lines:
  - the lines through the given point that meet the given line; and
  - the line through the given point that is parallel to the given line.

- Given a plane, and given a line that is contained in the plane or is parallel to it, the plane is equal to the union of those lines that lie in the plane and are parallel to the given line.