NOTES ON PERSPECTIVE GEOMETRY

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In standard three dimensional Euclidean space (which we can identify with $\mathbb{R}^3$), we fix a point O, and we consider *central projection* with respect to O.

Fix two planes, W and P, not through O, and not parallel to each other. Central projection gives a correspondence between them: a point on W corresponds to a point on P iff the two points lie on the same line through O.

We can think of O as the eye of an artist, of W as the "world plane" (for example, the ground), and of P as the "picture plane". Given a point on the world plane W, for example, a tiny flower on the ground, the corresponding point on the picture plane P (the flower in the picture) is where the "line of sight" from the flower to the artist's eye meets P.

Alternatively, we can think of O as lamp, of P as a plane of objects, and of W as a plane of shadows. For example, suppose that we have a lampshade with a circular metal rim. The lines that connect O to points on the rim sweep a cone through O. The shadow of the rim on a wall W is where these lines hit the wall. Namely it's the intersection of the cone with a plane W. So it's a conic section: a circle or ellipse, or a parabola, or a hyperbola.

Consider a picture of rail tracks. In the world plane W, a rail is straight. In the picture plane P, the picture of the rail is straight, and as we follow a point on the actual rail that moves very far away, its image in the picture plane typically approaches a "vanishing point" that does not correspond to any actual point on the world plane W.

The lines that connect points of the rail to O sweep out a "plane of lines of sight"; that's the plane that is determined by the line of the rail and by the point O. In that plane, there is exactly one line that does not pass through the rail: it is the line through O that is parallel to the rail. Lines of sight that connect O to farther and farther points on the rail approach this special line. This special line, although it lies on the "swept plane of lines of sight" determined by the rail, does not correspond to any actual point on the rail. (We declare it to correspond to the *point at infinity* of the rail.) The *vanishing point* is the intersection of this special line with the picture plane P. This point lies on the line on the picture plane that contains the picture of the rail, but it doesn't correspond to any actual point on the rail.

A point on the picture plane P does not correspond to any point on the world plane W exactly if it lies on a line through O that never meets W. That is, the points of P that do not correspond to any point on W are those points of P that lie on lines through O that are parallel to W. The lines on O that are parallel to W sweep the plane through O that is parallel to W. The points on P that do not correspond to any point on W are exactly those points of P that also lie on this plane. Thus, these are the points of intersection of the plane

P with the plane through O that is parallel to W. Typically, these two planes intersect in a line. This line is the *vanishing line* in the picture plane P, also known as the *horizon*.

The vanishing line in P is the intersection of P with the plane through O that is parallel to W. Similarly, the vanishing line in W is the intersection of W with the plane through O that is parallel to P; it consists of the points on the W that do not correspond to any point of P. These lines are both parallel to the intersection of W and P. It follows that they are parallel to each other. To summarize: the vanishing line in P is parallel to the vanishing line in W, and they are both parallel to the intersection of W and P.

For any straight line in W, its vanishing point in P lies on the horizon, because it doesn't correspond to any point of W. Conversely, every point on the horizon in P is a vanishing point for some line in W; in fact it is the vanishing point for many lines in W. Indeed, fix a point on the horizon. The line that connects it to O is parallel to W. So we can sweep W by lines that are parallel to it and lie in W. These lines in W all correspond to the given vanishing point on the horizon in P.

In some situations we don't have a vanishing point; the rail in the picture is parallel to the horizon in the picture. This happens if the special line in the "swept plane of lines of sight" – the line through O that is parallel to the rail – does not meet the picture plane P. That is, it happens when this line is parallel to the picture plane P. But this line is also parallel to W. So this happens when this line, and hence also the rail, is parallel to the intersection of P and W.

We have the following partly-defined maps.

$$\{\text{points in } P\} \xleftarrow{L\cap P \leftarrow\!\shortmid L} \{\text{lines through } O\} \xrightarrow{L\mapsto L\cap W} \{\text{points in } W\}$$

Each of these maps is one-to-one and onto but is not everywhere defined. The map to P is undefined for lines through O that are parallel to W; the images of these lines under the map to P form the vanishing line in W. The map to W is undefined for lines through O that are parallel to P; the images of these lines under the map to P form the vanishing line in P. The inverse of the map to W, composed by the map to P, gives the bijection from W minus its vanishing line to P minus its vanishing line that we get from central projection with respect to O.

To get maps that are everywhere defined, we need to add "points at infinity" to W and to P, yielding the *projective completions* of W and P. After doing this, we get maps

$$\text{the projective completion of } P \longleftarrow \{\text{lines through } O\} \longrightarrow \text{the projective completion of } W$$

that are everywhere defined and are bijections. The inverse of the second map composed with the first map gives a bijection from the projective completion of W to the projective completion of P that sends the points at infinity of W to the vanishing line in P (and the vanishing line in W to the points at infinity of P).