Today's goals are

(1) to define isotropy representations and the adjoint and coadjoint actions. These are relevant for the problem set.
(2) To being studying compact Lie groups, in preparation for "local normal forms".

Isotropy representation. Let G act on a manifold N, and let x be a fixed point for the action. For every $a \in G$, the map $a\colon N \to N$ takes x to itself, so its differential at x is a linear operator $a_*\colon T_xN \to T_xN$. In this way we get a linear G action on T_xN.

More generally, if G acts on N and x is an arbitrary point of N, we get a linear action of $H := \mathrm{Stab}(x)$ on T_xN. This action is called the *isotropy representation* at x.

Remark 0.1. Some people also call the stabilizer of x the "isotropy group" at x.

Adjoint and coadjoint representations. Let a Lie group G act on itself by conjugation:

$$a\colon g \mapsto aga^{-1}.$$

The point $g = 1$ is a fixed point for this action. The istropy action of G on $T_1G = \mathfrak{g}$ is called the *Adjoint action*, or *Adjoint representation*. It is denoted Ad.

The *coadjoint action*, or *coadjoint representation*, is the dual representation of G on $\mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \mathbb{R})$: for $a \in G$ and $\varphi \in \mathfrak{g}^*$,

$$\left(\mathrm{Ad}^*(a)\varphi\right)(\xi) = \varphi\left(\mathrm{Ad}^*(a^{-1})\xi\right)$$

for all $\xi \in \mathfrak{g}$.

Example 0.2. Suppose $G \subset \mathrm{GL}(n)$, so $\mathfrak{g} \subset M_{n,n}$. From

$$a(\exp t\xi)a^{-1} = a(I + t\xi + O(t^2))a^{-1} = I + t(a\xi a^{-1}) + O(t^2)$$

we deduce that the Adjoint action is conjugation:

$$\mathrm{Ad}(a)(\xi) = a\xi a^{-1}.$$

Invariant measures on compact Lie groups.

Theorem 0.3. *Let G be a compact Lie group. Then there exists a probability measure on G that is invariant under left and right translations and under inversion.*

Remark 0.4. On a locally compact topological group, a *Haar measure* is a regular Borel measure that is left-invariant. Fact: it exists and it is unique up to a positive scalar. But it might not be right-invariant.

If the compact Lie group G is connected, to prove Theorem 0.3 we take a left-invariant volume form and prove that it is also right-invariant and invariant under inversion. If G is not connected, to overcome issues of orientation, we will work with densities instead of volume forms.

Remark on orientations. A Lie group is parallelizable:

$$TG \cong G \times \mathfrak{g} \text{ via left translation: } a \cdot \xi \longleftrightarrow (a, \xi),$$

hence orientable.

- If G is connected, then for every $a \in G$ the left and right translations L_a and R_a can be deformed through diffeomorphisms of G to the identity map on G. So they preserve orientation.
- A disconnected example:

  $$G = \mathbb{Z}_2 \ltimes K$$

  where K is a torus of odd dimension (e.g. S^1) and where the non-trivial element of $\mathbb{Z}_2$ acts on K by inversion $g \to g^{-1}$, which reverses orientation. (Inversion is an automorphism of K because K is abelian.) I.E.,

  $$(\epsilon_1, k_1) \cdot (\epsilon_2, k_2) = \begin{cases} (\epsilon_1\epsilon_2, k_1 k_2) & \text{if } \epsilon_2 = 1 \\ (\epsilon_1\epsilon_2, k_1^{-1} k_2) & \text{if } \epsilon_2 = -1. \end{cases}$$

  Because conjugation by the non-zero element ϵ of $\mathbb{Z}_2$ does not preserve orientation, L_ϵ and R_ϵ cannot both preserve orientation.

A lemma about orientations.

Lemma 0.5. *Let G be a Lie group. Pick an orientation on the Lie algebra $\mathfrak{g}$. Equip G with the orientation induced from the left trivialization*

$$\begin{array}{ccc} TG & \cong & G \times \mathfrak{g} \\ a \cdot \xi & \longleftrightarrow & (a, \xi). \end{array}$$

Fix $b \in G$. Then

$$R_b \colon G \to G$$

either preserves orientation everywhere or reverses orientation everywhere.

Proof. Because left and right translations commute, we have a commuting diagram

$$\begin{tikzcd} \text{neighbourhood of } g \arrow[r, "R_b"] & \text{neighbourhood of } gb \\ \text{neighbourhood of } 1 \arrow[r, "R_b"] \arrow[u, "L_g"] & \text{neighbourhood of } b \arrow[u, "L_g"'] \end{tikzcd}$$

Because left translations L_g preserve orientation, if $R_b|_1$ preserves orientation then $R_b|_g$ preserves orientation for all g, and if $R_b|_1$ reverses orientation then $R_b|_g$ reverses orientation for all g. $\square$

Volume forms and densities. Recall that a *volume form* is a non-vanishing section of $\bigwedge^{\text{top}} T^*G$, and that $\bigwedge^{\text{top}} T^*G$ is a rank one real line bundle. A volume form vol determines an *orientation.* Integration with respect to this orientation gives a positive measure. Because G is compact, its total measure is finite, so we can rescale to get a *probability* measure.

This measure can also be described as the integration of a *density*, which we denote $|\operatorname{vol}|$. A density is a section of the tensor of $\bigwedge^{\text{top}} T^*M \otimes L$ where L is the orientation line bundle.

(The orientation line bundle L is a real line bundle that can be constructed in the following way. Take an atlas $\{\varphi_i \colon U_i \to \Omega_i\}$ for M. Define $\psi_{ij} \colon U_i \cap U_j \to \{1, -1\}$ by the signs of the Jacobi determinants of the transition maps $\varphi_j \circ \varphi_i^{-1}$. Obtain L by gluing the trivial bundles $U_i \times \mathbb{R}$ with the functions ψ_{ij}.)

The bundle $\bigwedge^{\text{top}} T^*M \otimes L$ is trivializable, but there is no natural trivialization. A section of this bundle over an open subset U of M can be viewed as an equivalence class of pairs (α, o), where α is a top degree form on U and o is an orientation of o, under the equivalence $(\alpha, o) \sim (-\alpha, -o)$. A compactly supported section of this bundle can be integrated: the integral of (α, o) is the integral of the form α with respect to the orientation o. The answer is independent of choices of orientations.

Finally – a density can be pulled back through a diffeomorphism (but not through an arbitrary smooth map). Such a pullback does not effect its integral.

A density is a special case of a “differential form of odd type” as described in de Rham's 1955 book “Differentiable Manifolds” (English translation: 1984, Springer-Verlag).

Proof of Theorem 0.3. Pick $0 \neq v \in \bigwedge^{\text{top}} \mathfrak{g}^*$. Define vol to be the left invariant volume form on G with $\operatorname{vol}|_1 = v$. I.E., $\operatorname{vol}|_a = (L_{a^{-1}})^* v$.

Claim: Let $b \in G$. Then $R_b^* \operatorname{vol} = \pm \operatorname{vol}$.

Proof of the claim: Because $\dim \bigwedge^{\text{top}} \mathfrak{g}^* = 1$, there exists a real number λ such that $(R_b^* \operatorname{vol})|_1 = \lambda \operatorname{vol}|_1$.

$$L_a^*(R_b^* \operatorname{vol}) = R_b^*(L_a^* \operatorname{vol}) = R_b^* \operatorname{vol}$$

because L_a and R_b commute and because $L_a^* \operatorname{vol} = \operatorname{vol}$. So $R_b^* \operatorname{vol}$ is left invariant.

Because $R_b^* \operatorname{vol}$ and $\lambda \operatorname{vol}$ are left invariant and coincide at $g = 1$, they coincide everywhere:

$$R_b^* \operatorname{vol} = \lambda \operatorname{vol}.$$

We now integrate these forms with respect to the left invariant orientation induced from vol:

$$\int_G R_b^* \operatorname{vol} = \lambda \int_G \operatorname{vol}.$$

By the lemma on orientations, the diffeomorphism R_b either preserves orientation everywhere or reverses orientation everywhere. The left hand side is equal to vol or $-\operatorname{vol}$ according to these two cases. We deduce that $\lambda = \pm 1$. So $R_b^* \operatorname{vol} = \pm \operatorname{vol}$, and $R_b^* |\operatorname{vol}| = |R_b^* \operatorname{vol}| = |\operatorname{vol}|$.

Claim: $|\operatorname{vol}|$ is invariant under inversion

$$\begin{aligned} I \colon G &\to G \\ g &\mapsto g^{-1}. \end{aligned}$$

Proof of the claim: Because $|\operatorname{vol}|$ is right invariant, $I^* \operatorname{vol}$ is left invariant:

$$L_a^* I^* |\operatorname{vol}| = (I L_a)^* |\operatorname{vol}| = (R_{a^{-1}} I)^* |\operatorname{vol}| = I^* (R_{a^{-1}}^* |\operatorname{vol}|) = I^* |\operatorname{vol}|.$$

Let $\lambda > 0$ be such that $I^* |\operatorname{vol}|_1 = \lambda |\operatorname{vol}|_1$. Then $I^* |\operatorname{vol}|$ and $\lambda |\operatorname{vol}|$ are left invariant densities that coincide at 1, so they are equal: $I^* |\operatorname{vol}| = \lambda |\operatorname{vol}|$. Integrating both sides over G, we deduce that $\lambda = 1$.

*** Can we also prove Theorem 0.3 by taking an invariant volume form on the identity component G_0 of G and using the fact that G/G_0 is a finite group? ***

Averaging. Until now, we integrated volume forms and densities over G. Now that we have established the existence of an invariant probability measure on G, we can use it to integrate *functions* over G. This enables us to apply *averaging* arguments. We give two examples of averaging.

Invariant inner products.

Lemma 0.6. *Let a compact Lie group G act linearly on a finite dimensional real vector space V. Then there exists on V a G invariant inner product $\langle \cdot, \cdot \rangle$.*

Proof. Start with any inner product, $\langle \cdot, \cdot \rangle'$. Then define, for $u, v \in V$,

$$\langle u, v \rangle = \int_G \langle a \cdot u, a \cdot v \rangle' \, da$$

where da is an invariant probability measure on G. □

A (Hermitian) inner product on a *complex* vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ that is linear with respect to the second entry, satisfies $\langle u, v \rangle = \overline{\langle v, u \rangle}$, and is positive definite: $\langle u, u \rangle > 0$ whenever $u \neq 0$. On a complex vector space V, a Hermitian inner product always exists. (Compose a linear isomorphism $V \cong \mathbb{C}^n$ with the standard Hermitian inner product $(z, w) \mapsto \sum \overline{z}_j w_j$.) An averaging argument as above shows that, if G acts on V through complex linear transformations, then there exists on V a G *invariant* Hermitian inner product.

The complex of invariant forms. Let a Lie group G act on a manifold M. If a differential form α is G-invariant, so is $d\alpha$. Thus, the invariant differential forms define a subcomplex of the complex of all differential forms. We denote these complexes by

$$\left(\Omega^{\cdot}(M)^G, d\right) \hookrightarrow (\Omega^{\cdot}(M), d)\,. \tag{0.7}$$

Recall that the de Rham cohomology $H^{\cdot}_{\mathrm{dR}}(M)$ is defined to be the cohomology of the complex $(\Omega^{\cdot}(M), d)$.

Suppose that G is connected. For every $a \in G$, because the transformation $\rho(a) \colon M \to M$ is homotopic to the identity map (through the maps $\rho(a_t)$, where $t \mapsto a_t$ is a path in G from a to 1), it induces the identity map on cohomology.

Lemma 0.8. *Let a compact connected Lie group act on a manifold M. Then the inclusion of complexes* (0.7) *induces an isomorphism in cohomology.*

In class we didn't get to the proof, but here it is:

Proof. Let α' is a closed k-form on M. Let

$$\alpha = \int_{g\in G} (g^*\alpha')dg.$$

Then α is closed and G invariant, and it is in the same cohomology class as α'. This shows that (0.7) induces a *surjection* in cohomology.

Let α be an invariant k-form on M. Suppose that α is exact. Let β' be a primitive: $\alpha = d\beta'$. Let

$$\beta := \int_{g\in G} (g^*\beta')dg.$$

Then β is G invariant, and

$$d\beta = \int_{g\in G} (d(g^*\beta'))\,dg = \int_{g\in G} (g^*d\beta') = \int_{g\in G} \alpha dg = \alpha.$$

("dg" refers to the invariant measure on G, whereas in $d(g^*\beta')$ the symbol "d" denotes the exterior derivative.) This shows that (0.7) induces an *injection* in cohomology. $\square$