MAT 133Y1Y TERM TEST 1

Thursday, 9 June, 2011, 10:30 am – 12:30 pm

Code 1

FAMILY NAME ______________ Solutions

GIVEN NAME(S) ______________

STUDENT NO. ______________

SIGNATURE ______________

GRADER'S REPORT	
Question	Mark
MC/40	
B1/15	
B2/15	
B3/15	
B4/15	
TOTAL	

NOTE:

1. **Aids Allowed:** Calculator with empty memory, to be supplied by student. Absolutely no graphing calculators allowed.
2. **Instructions:** Fill in the information on this page and ensure that the test contains 10 pages.
3. This test has 10 multiple choice questions worth 4 marks each and 4 written-answer questions worth 15 marks each.

   For the **multiple choice questions** indicate your answers by circling the appropriate letters (**A**, **B**, **C**, **D**, or **E**) on **this page (page 1)**. A multiple choice question left blank on **this page**, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the **written-answer questions**, present your solutions in the spaces provided. Use the multiple choice question pages or the back of any of the pages for rough work, for any of the questions.

ANSWER BOX FOR PART A

Circle the correct answer.

1.	**A**	**B**	**C**	**D**	**E**
2.	**A**	**B**	**C**	**D**	**E**
3.	**A**	**B**	**C**	**D**	**E**
4.	**A**	**B**	**C**	**D**	**E**
5.	**A**	**B**	**C**	**D**	**E**
6.	**A**	**B**	**C**	**D**	**E**
7.	**A**	**B**	**C**	**D**	**E**
8.	**A**	**B**	**C**	**D**	**E**
9.	**A**	**B**	**C**	**D**	**E**
10.	**A**	**B**	**C**	**D**	**E**

PART A. Multiple Choice

1. *[4 marks]*

   What nominal rate compounded quarterly is most nearly equivalent to 5% compounded daily? (1 year = 365 days.)

   A 5.125%

   B 5.000%

   C 5.017%

   D 5.031%

   E 5.093%

Let r = the unknown rate (compounded quarterly)

1 quarter $= \frac{1}{4}$ year $= \frac{365}{4}$ days

$$1+\frac{r}{4} = \left(1+\frac{.05}{365}\right)^{\frac{365}{4}}$$

(Both sides of this last equality represent the ratio by which money multiplies in 1 quarter.)

$$\text{So } r = 4\left(\left(1+\frac{.05}{365}\right)^{\frac{365}{4}} - 1\right)$$

2. *[4 marks]*

   A loan of \$5000 is to be repaid with payments of $\$X$ two years from now and $\$2X$ four years from now, plus a final payment of \$1000 five years from now. If the effective annual interest rate of the loan is 4%, then $\$X =$

   A \$1586.11

   B \$1605.74

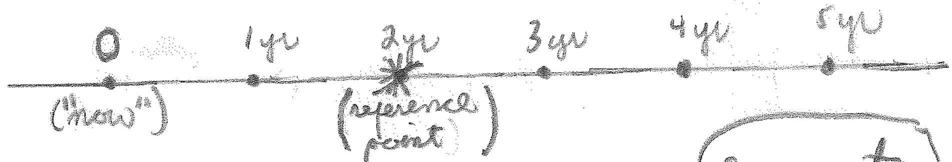
   C \$1487.90

   D \$1455.83

   E \$1523.46

debt

$\$5000 \longrightarrow \$5000(1.04)^2$



payments

$\$X$

$\$2X(1.04)^{-2} \longleftarrow \$2X$

$\$1000(1.04)^{-3} \longleftarrow \$1000$

Equation of value: $\$X + \$2X(1.04)^{-2} + \$1000(1.04)^{-3} = \$5000(1.04)^2$

Solving for $\$X$: $\$X = \dfrac{5000(1.04)^2 - 1000(1.04)^{-3}}{1+2(1.04)^{-2}}$

3. *[4 marks]*

If interest is compounded continuously, what nominal annual rate (to the nearest 0.01%) is required for money to double in 10 years?

- **A** 6.93%
- **B** 6.15%
- **C** 7.24%
- **D** 6.47%
- **E** 6.34%

Let r = the unknown rate (compounded continuously)

Then for any principal \$P, $\$Pe^{10r} = \$2P$

(Both sides of this equality represent the future value (in 10 years) of \$P.)

Solving for r:

$$e^{10r} = 2$$
$$10r = \ln 2$$
$$r = \frac{\ln 2}{10} = 10 \ln 2 \,\%$$

4. *[4 marks]*

If annual deposits of \$5000 each are made in an account earning interest at 6% compounded annually, then the amount in the account just before the 13^{th} deposit will be

- **A** \$84,410.69
- **B** \$89,349.71
- **C** \$89,410.69
- **D** \$94,410.69
- **E** \$84,349.71

Just after the 13^{th} deposit, the account will have

$$\$5000\, s_{\overline{13}|.06} = \$94{,}410.69$$

Just before the 13^{th} deposit, it will have \$5000 less.

5. *[4 marks]*

The price of a $\$100$ bond which has 8 years to maturity, semiannual interest payments at an annual coupon rate of 5%, and an annual yield rate of 4.5%, is

A $105.87

B $100.00

C $96.91

D $107.25

E $103.33

$$\$100\,(1.0225)^{-16} + \$2.50\, a_{\overline{16}|\,1.0225}$$

(Each coupon is $2.50)

6. *[4 marks]*

If $A = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $(7I - 2AB^T)^2$

A $= \begin{bmatrix} 24 & 32 \\ -7 & -19 \end{bmatrix}$

B $= \begin{bmatrix} -18 & 15 \\ -30 & 64 \end{bmatrix}$

C $= \begin{bmatrix} 33 & -48 \\ -8 & 25 \end{bmatrix}$

D $= \begin{bmatrix} 42 & -5 \\ 6 & 35 \end{bmatrix}$

E is not defined

$$AB^T = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$$

$$2AB^T = \begin{bmatrix} 4 & 12 \\ 2 & 6 \end{bmatrix}$$

$$7I - 2AB^T = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 4 & 12 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -12 \\ -2 & 1 \end{bmatrix}$$

$$(7I - 2AB^T)^2 = \begin{bmatrix} 3 & -12 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & -12 \\ -2 & 1 \end{bmatrix}$$

7. *[4 marks]*

Let $n = 1, 2, \ldots$ be fixed and let I denote the $n \times n$ identity matrix. If A is an $n \times n$ matrix such that $(3A + I)^2 = 4I$, then A^{-1}

A does not exist

B $= \frac{1}{2}A + \frac{3}{2}I$

C $= 2A + 3I$

D $= 3A + 2I$

E $= \frac{3}{2}A + \frac{1}{2}I$

$$9A^2 + 6A + I = 4I$$

$$9A^2 + 6A = 3I$$

$$3A^2 + 2A = I$$

$$A(3A + 2I) = I$$

8. *[4 marks]*

The matrix equation $AX = B$, where $A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 2 & 1 & -3 \\ 3 & 1 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, has solution

A $x = 6,\ y = -1,\ z = 1$

B $x = 1,\ y = -5,\ z = -1$

C $x = -1,\ y = 1,\ z = -7$

D $x = 1,\ y = 4,\ z = 1$

E $x = -8,\ y = -1,\ z = -1$

If $X = A^{-1}B$

then $AX = B$

9. *[4 marks]*

If a homogeneous system of 3 linear equations has 7 unknowns, then the solution set of the system has

A at most 3 parameters

B exactly 3 parameters

C at least 4 parameters

D at most 4 parameters

E exactly 4 parameters

Row reduction will lead to an augmented matrix having 3 rows with no row of zeros, in which case the solution will give the values of exactly 3 variables in terms of exactly 4 others (the parameters). Or the final (row-reduced) matrix will have at least one row of zeros, and the solution gives < 3 variables in terms of > 4 parameters.

10. *[4 marks]*

The system $\begin{array}{rcrcrcl} x & & & + & z & = & 4 \\ -x & + & y & + & z & = & 5 \\ x & + & y & + & az & = & 6 \end{array}$ has no solution if the constant a equals

A 4

B 1

C 5

D 2

E 3

Row reduction:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ -1 & 1 & 1 & 5 \\ 1 & 1 & a & 6 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 2 & 9 \\ 0 & 1 & a-1 & 2 \end{array}\right]$$

$$\simeq \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & a-3 & -7 \end{array}\right]$$

and if $a=3$ the third row represents the equation $0=-7$.

(Conversely, if $a \neq 3$, then $z = \frac{7}{3-a}$, $y = 9-2z$, $x = 4-z$.)

PART B. Written-Answer Questions
SHOW YOUR WORK.

B1.(a) *[5 marks]*

Susan makes equal monthly deposits into an account earning 6% compounded monthly so that the account will have \$50,000 immediately after her 96$^{\text{th}}$ deposit. Assuming interest will always be 6% compounded monthly, what should be the amount of each deposit?

$$\frac{\$50{,}000}{s_{\overline{96}|.005}} = \frac{\$250}{(1.005)^{96}-1} = \$407.07$$

B1.(b) *[10 marks]*

Contrary to the assumption in question B1.(a), just after Susan's 48$^{\text{th}}$ deposit her account's interest rate changes to 4.8% compounded monthly. Determine the <u>new</u> monthly deposit she must make during the remaining 4 years in order to accumulate \$50,000 immediately after a total of 96 deposits.

If a time just after the 48$^{\text{th}}$ deposit is used as reference point (designated $*$), the resulting equation of value for this problem is:

present value (at $*$) of the last 48 deposits + amount accumulated (at $*$) = present value (at $*$) of \$50,000

If \$R = the new monthly deposit (after $*$), then

$$\$R\, a_{\overline{48}|.004} + \$50{,}000 \frac{s_{\overline{48}|.005}}{s_{\overline{96}|.005}} = \$50{,}000(1.004)^{-48}$$

Solving, $\$R = \frac{\$200}{1-(1.004)^{-48}}\left((1.004)^{-48} - \frac{1}{(1.005)^{48}+1}\right) = \$441.79$

B2. *[15 marks]*

A \$400,000, 15 year mortgage is to be repaid with equal monthly payments. Interest is 6% compounded semiannually.

B2.(a) *[7 marks]*

What is the amount of each payment?

Let r = effective monthly interest ; $\underline{\underline{1+r = (1.03)^{\frac{1}{6}}}}$

Each payment is $\dfrac{\$400{,}000}{a_{\overline{180|}r}}$

$= \$400{,}000 \dfrac{(1.03)^{\frac{1}{6}}-1}{1-(1.03)^{-30}} = \$3359.53$

B2.(b) *[8 marks]*

How much principal is repaid in the 120^{th} payment of the mortgage?

Let \$R = the amount of each payment.

Just after the 119^{th} payment, 61 payments remain, principal outstanding = $\$R\, a_{\overline{61|}r}$,

interest in the 120^{th} payment = $\$R\, a_{\overline{61|}r} \cdot r$,

and principal repaid = $\$R - \$R\, a_{\overline{61|}r} \cdot r$

$= \$R\left(1-\left(1-(1+r)^{-61}\right)\right) = \$R\,(1+r)^{-61}$

$= \$R\,(1.03)^{-\frac{61}{6}} = \$3359.53 \cdot (1.03)^{-\frac{61}{6}}$

$= \$2487.52$

B3.(a) *[8 marks]*

Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 1 \\ -2 & -5 & -1 \\ 2 & 1 & 3 \end{bmatrix}$, or show that it has no inverse.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -2 & -5 & -1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \simeq \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 2 & 1 & 0 \\ 0 & -3 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\simeq \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 5 & 2 & 0 \\ 0 & 1 & -1 & -2 & -1 & 0 \\ 0 & 0 & -2 & -8 & -3 & 1 \end{array}\right] \simeq \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & -\frac{5}{2} & \frac{3}{2} \\ 0 & 1 & 0 & 2 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 4 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$$

B3.(b) *[7 marks]*

Find all solutions of the system

$$\begin{aligned} w + 2x + 3y - z &= -3 \\ 4w + 3x + 2y + z &= -2 \\ 3w + 2x + y + 3z &= 3 \end{aligned}.$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & -3 \\ 4 & 3 & 2 & 1 & -2 \\ 3 & 2 & 1 & 3 & 3 \end{array}\right] \simeq \left[\begin{array}{cccc|c} 1 & 2 & 3 & -1 & -3 \\ 0 & -5 & -10 & 5 & 10 \\ 0 & -4 & -8 & 6 & 12 \end{array}\right]$$

$$\simeq \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 2 & -1 & -2 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right] \simeq \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

The last matrix is in reduced row-echelon form and represents the system $w - y = -1$, $x + 2y = 0$, $z = 2$. This yields the solution set, $w = y - 1$, $x = -2y$, $z = 2$, where $y =$ any real.

B4. *[15 marks]*

A financial advisor has three clients whose investments are in bonds, stocks, and financial reserves. The fraction of each client's total investment in each of the three mediums is given by the table below.

	Client #1	Client #2	Client #3
Bonds	0.4	0.3	0.6
Stocks	0.4	0.6	0.3
Cash	0.2	0.1	0.1

If the total investments (in 1000's of dollars) in each of bonds, stocks, and cash reserves are 166 , 142 , and 62 , respectively, find the amount that each client has invested. **Show your work.**

Let x_i = client # i's total investment $(i=1,2,3)$.

These are determined by the system

$$\begin{bmatrix} .4 & .3 & .6 \\ .4 & .6 & .3 \\ .2 & .1 & .1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 166 \\ 142 \\ 62 \end{bmatrix}\begin{matrix} \leftarrow \text{bonds} \\ \leftarrow \text{stocks} \\ \leftarrow \text{cash} \end{matrix}$$

Solution by row reduction:

$$\left[\begin{array}{ccc|c} .4 & .3 & .6 & 166 \\ .4 & .6 & .3 & 142 \\ .2 & .1 & .1 & 62 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & .75 & 1.5 & 415 \\ 0 & .3 & -.3 & -24 \\ 0 & -.05 & -.2 & -21 \end{array}\right]$$

$$\simeq \left[\begin{array}{ccc|c} 1 & 0 & 2.25 & 475 \\ 0 & 1 & -1 & -80 \\ 0 & 0 & -.25 & -25 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 0 & 0 & 250 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 100 \end{array}\right]$$

$x_1 = 250$ $\quad$ $x_2 = 20$ $\quad$ $x_3 = 100$

(1000's of \$)