MAT 133Y1Y TERM TEST 1

Thursday, 4 June, 2009, 10:30 am – 12:30 pm

Code 1

FAMILY NAME ______________________ Solutions

GIVEN NAME(S) ______________________

STUDENT NO. ______________________

SIGNATURE ______________________

GRADER'S REPORT	
Question	Mark
MC/40	
B1/15	
B2/15	
B3/15	
B4/15	
TOTAL	

NOTE:

1. **Aids Allowed:** Calculator with empty memory, to be supplied by student. Absolutely no graphing calculators allowed.
2. **Instructions:** Fill in the information on this page and ensure that the test contains 11 pages.
3. This test has 10 multiple choice questions worth 4 marks each and 4 written-answer questions worth 15 marks each.

   For the **multiple choice questions** indicate your answers by circling the appropriate letters (**A**, **B**, **C**, **D**, or **E**) on **this page (page 1)**. A multiple choice question left blank on **this page**, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the **written-answer questions**, present your solutions in the spaces provided. Use the multiple choice question pages or the back of any of the pages for rough work, for any of the questions.

ANSWER BOX FOR PART A

Circle the correct answer.

1.	**A**	**B**	**C**	**D**	**E**
2.	**A**	**B**	**C**	**D**	**E**
3.	**A**	**B**	**C**	**D**	**E**
4.	**A**	**B**	**C**	**D**	**E**
5.	**A**	**B**	**C**	**D**	**E**
6.	**A**	**B**	**C**	**D**	**E**
7.	**A**	**B**	**C**	**D**	**E**
8.	**A**	**B**	**C**	**D**	**E**
9.	**A**	**B**	**C**	**D**	**E**
10.	**A**	**B**	**C**	**D**	**E**

PART A. Multiple Choice

1. *[4 marks]*

   What nominal annual rate compounded semiannually is most nearly equivalent to 7% per year compounded continuously?

   A 7.06%

   B 7.12%

   C 7.02%

   D 7.00%

   E 7.15%

In 1 year, money is multiplied by the factor $e^{0.07}$, and in $\frac{1}{2}$ year, effectively by the factor $e^{0.035}$. Effective rate over $\frac{1}{2}$ year is $e^{0.035}-1$ and nominal annual rate is $2(e^{0.035}-1)$.

2. *[4 marks]*

   A loan of $\$20,000$ is to be repaid with payments of $\$X$ one year from now and $\$3X$ two years from now. If the effective annual rate of the loan is 8%, then $X =$

   A 5717.65

   B 5523.09

   C 6422.48

   D 6175.28

   E 5447.53

\$20,000

0 yr — 1 yr — 2 yr

now

$\$X(1.08)^{-1} \longleftarrow \$X$

$\$3X(1.08)^{-2} \longleftarrow \$3X$

$X(1.08)^{-1} + 3X(1.08)^{-2} = 20,000$ has solution $X = \dfrac{20,000}{(1.08)^{-1}+3(1.08)^{-2}}$.

3. *[4 marks]*

For one year, monthly deposits of $100 each are made into an account which earns 12% compounded monthly. If no further deposits or withdrawals are made, how much money (to the nearest dollar) will the account have, one year after the last deposit?

A $1171

B $1429

C $1556

D $1268

E $1302

$$\$100\, s_{\overline{12}|.01}(1.01)^{12} = \$100\,\frac{(1.01)^{12}-1}{.01}(1.01)^{12}$$

$$= \$10{,}000\left((1.01)^{12}-1\right)(1.01)^{12}.$$

4. *[4 marks]*

A certain ordinary annuity costs $233,791.74 and makes annual payments of $20,000 each. If it earns interest at 5% compounded annually, how many payments does it make?

A 19

B 15

C 17

D 16

E 18

Let n = the number of payments.

The present value of the annuity is

$$\$233{,}791.74 = \$20{,}000\,\frac{1-(1.05)^{-n}}{.05}$$

To solve this for n,

$$233{,}791.74 = 400{,}000\left(1-(1.05)^{-n}\right)$$

$$.58447935 = 1-(1.05)^{-n}$$

$$(1.05)^{-n} = .41552065$$

$$-n\ln(1.05) = \ln(.41552065) \text{ so } n = \frac{-\ln(.41552065)}{\ln(1.05)}$$

5. *[4 marks]*

To purchase a $\$450,000$ house a person pays $\$50,000$ down and takes on a 25 year mortgage with monthly payments and interest at 6% compounded semiannually. The monthly mortgage payments will be closest to

A $2899

B $2167

C $2577

D $2559

E $2599

Amount owing = $\$400,000$

Monthly rate = $(1.03)^{\frac{1}{6}}-1$

Let $\$R$ denote the monthly payment.

$\$R\, a_{\overline{300}|(1.03)^{\frac{1}{6}}-1} = \$400,000$, so

$$\$R = \$400,000 \, \frac{(1.03)^{\frac{1}{6}}-1}{1-(1.03)^{-50}}$$

6. *[4 marks]*

Let A denote a 5×4 matrix and let B denote a 4×4 matrix. Which of the following is **not** defined?

A AB

B BA^T

C A^TA

D A^2

E B^2

A is not square!

7. *[4 marks]*

If $n = 1, 2, \ldots$ is fixed, O denotes the $n \times n$ zero matrix, I denotes the $n \times n$ identity matrix, and A denotes an $n \times n$ matrix such that $(2A - I)^2 = O$, then $A^{-1} =$

A $A - 2I$

B $4(I - A)$

C $2(I - A)$

D $2I - A$

E $4(A - I)$

$$4A^2 - 4A + I = O$$

$$I = 4A - 4A^2$$

$$I = 4(I - A)A$$

8. *[4 marks]*

What are the **only** real values of a and b for which the following matrix is in reduced row-echelon form?

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & a & b & 0 \end{bmatrix}$$

A $a = 0, b = 0$ and $a = 0, b = 1$ and $a = 1, b = 0$

B $a = 1, b = 0$ and $a = 0, b =$ any real number

C $a = 0, b = 0$ and $a = 0, b = 1$ and $a = 1, b =$ any real number

D $a = 0, b = 0$ and $a = 1, b = 0$

E $a = 0, b = 0$ and $a = 1, b =$ any real number

Not A, B, or C: with $a = 0, b = 1$ the matrix $\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ is in row-echelon form but not reduced row-echelon form, as the third column has a leading element and a second non-zero element.

9. *[4 marks]*

If $\begin{bmatrix} 2 & -3 \\ 8 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$, what is the value of x?

Note that $\begin{bmatrix} 13 & 3 \\ -8 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 8 & 13 \end{bmatrix} = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix}$.

A $x = -8$

B $x = 35$

C $x = 38$

D $x = 24$

E $x = 14$

$$\begin{bmatrix} 13 & 3 \\ -8 & 2 \end{bmatrix} \left(\begin{bmatrix} 2 & -3 \\ 8 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 13 & 3 \\ -8 & 2 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \end{bmatrix}$$

$$\left(\begin{bmatrix} 13 & 3 \\ -8 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 8 & 13 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1900 \\ -400 \end{bmatrix}.$$

$$\begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1900 \\ -400 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 38 \\ -8 \end{bmatrix}$$

10. *[4 marks]*

The system

$$\begin{array}{ccccccccccc} v & + & 2w & + & 5x & - & 3y & - & z & = & 0 \\ & & 3w & - & 2x & + & y & + & z & = & 0 \\ & & w & + & 2x & - & y & + & z & = & 0 \end{array}$$

has

A a three-parameter family of solutions

B only the trivial solution $v = w = x = y = z = 0$

C a one-parameter family of solutions

D a four-parameter family of solutions

E a two-parameter family of solutions

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & -2 \\ 0 & 1 & 2 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 & \frac{19}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & \frac{8}{3} \end{bmatrix}$$

$$\simeq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so the system can be solved for v, w, and x in terms of parameters y and z.

PART B. Written-Answer Questions
SHOW YOUR WORK.

B1. *[15 marks]*

Betty has won a lottery which will pay her \$4000 each month for 1 year, and then \$1000 each month for the next 4 years. She will receive her first \$1000 payment 1 month after her last \$4000 payment. If she invests each lottery cheque just after receiving it, at 4.2% compounded monthly, what will be the amount of her investment immediately after she deposits her last lottery payment?

Monthly interest $= \frac{4.2\%}{12} = .0035$

Method ① by computing the difference of two annuities that end at the same time:

$$\$4000\, s_{\overline{60}|.0035} - \$3000\, s_{\overline{48}|.0035}$$

$$= \$4000 \frac{(1.0035)^{60} - 1}{.0035} - \$3000 \frac{(1.0035)^{48} - 1}{.0035}$$

$$= \frac{\$1000}{.0035}\left(4(1.0035)^{60} - 3(1.0035)^{48} - 1\right)$$

$$= \$110{,}038.33$$

Method ② by treating the 2 types of cheque as 2 separate annuities that end at different times:

$$\$4000\, s_{\overline{12}|.0035}(1.0035)^{48} + \$1000\, s_{\overline{48}|.0035}$$

$$= \$110{,}038.33$$

B2. *[15 marks]*

Acme Income Fund issues a \$100 bond which matures in 20 years and has semiannual coupons with an annual coupon rate of 4.8%. Just 5 years after the bond is issued it is trading in the market at \$104.

B2.(a) *[10 marks]*

To achieve an accuracy within \$0.10 in the price of the bond, what is its current annual yield to maturity?

Let i denote the semiannual yield (so that the annual yield is $2i$). Each coupon is worth \$2.40 so i must satisfy

$$104 = \underbrace{100(1+i)^{-30} + 2.4\,\frac{1-(1+i)^{-30}}{i}}_{RHS}$$

(There are 15 more years until maturity.)

$i < .024$ because $104 > 100$. By trial and error with $i = .02$, RHS is 108.96 — too large (by about \$5) so i is maybe halfway between .020 and .024. With $i = .022$, RHS $= 104.36$, too large by 0.36. With $i = .023$, RHS $= 102.15$, 2.21 less than with $i = .022$.

$\frac{.36}{2.21} \approx .16 \approx .2$ so try $i = \underline{.0222}$. With $i = .0222$, RHS is 103.91, which is within the tolerance of the question, so $2i =$ 4.44% approximately.

(with $i = .02216$, RHS $= 104.001$ so a better estimate for annual yield is 4.43%.)

B2.(b) *[5 marks]*

On the same day (just 5 years after the bond is issued), Acme announces that at maturity it will pay \$110 to redeem the bond. Assuming that this announcement has no effect on the yield to maturity, exactly (to the nearest cent) what happens to the price of the bond?

Just before the announcement, the market price (\$104) was equal to

$$100(1.0222)^{-30} + 2.4\, a_{\overline{30}|.0222}$$

Just after the announcement (with the yield rate, coupon rate, and number of coupons remaining all unchanged), the market price changed to

$$110(1.0222)^{-30} + 2.4\, a_{\overline{30}|.0222}$$

Only the first term has changed. The market price increased by the amount

$$\$10(1.0222)^{-30} = \boxed{\$5.18}.$$

(With the better estimate, .02216, for semiannual yield, the price increase is still \$5.18, to the nearest cent.)

B3. *[15 marks]*

Find **all** solutions of
$$\begin{array}{rcrcrcr} x & + & y & + & 3z & = & 1 \\ 2x & - & 3y & + & z & = & -13 \\ 3x & + & 2y & + & 8z & = & 0 \\ 4x & + & 3y & + & 11z & = & 1 \end{array}.$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & -3 & 1 & -13 \\ 3 & 2 & 8 & 0 \\ 4 & 3 & 11 & 1 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & -5 & -5 & -15 \\ 0 & -1 & -1 & -3 \\ 0 & -1 & -1 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x = -2 - 2z$$
$$y = 3 - z$$

(z = any real)

B4. *[15 marks]*

The input-output matrix for an economy of 2 industries (**A** and **B**) is given below.

	A	B	final demand
A	14	16	40
B	14	16	10
other sources	42	8	

How much input from other sources will **A** require if the same economy is to satisfy the new consumer demands of 20 from **A** and 30 from **B**?

The A and B columns have respective totals 70 and 40 so this economy has technology matrix

$$\begin{array}{ccc} & A & B \\ A & \frac{14}{70} & \frac{16}{40} \\ B & \frac{14}{70} & \frac{16}{40} \\ \text{other} & \frac{42}{70} & \frac{8}{40} \end{array}, \text{ or more simply, } \begin{array}{ccc} & A & B \\ A & \frac{1}{5} & \frac{2}{5} \\ B & \frac{1}{5} & \frac{2}{5} \\ \text{other} & \frac{3}{5} & \frac{1}{5} \end{array}$$

If A and B are to satisfy the new consumer demands, their outputs (here denoted x_A and x_B) must satisfy

$$x_A = \frac{1}{5}x_A + \frac{2}{5}x_B + 20 \qquad \frac{4}{5}x_A - \frac{2}{5}x_B = 20$$
$$x_B = \frac{1}{5}x_A + \frac{2}{5}x_B + 30 \text{, that is, } -\frac{1}{5}x_A + \frac{3}{5}x_B = 30$$

To find x_A, $\left[\begin{array}{cc|c} \frac{4}{5} & -\frac{2}{5} & 20 \\ -\frac{1}{5} & \frac{3}{5} & 30 \end{array}\right] \approx \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 25 \\ 0 & \frac{1}{2} & 35 \end{array}\right] \approx \left[\begin{array}{cc|c} 1 & 0 & 60 \\ 0 & 1 & 70 \end{array}\right]$,

so that $x_A = 60$ and A now requires $\frac{3}{5} \times 60 = 36$ from other sources.