MAT 133Y1Y TERM TEST 2

Thursday, 8 July, 2010, 10:30 am – 12:30 pm

Code 1

FAMILY NAME Solutions

GIVEN NAME(S) __________

STUDENT NO. __________

SIGNATURE __________

GRADER'S REPORT	
Question	Mark
MC/40	
B1/15	
B2/15	
B3/15	
B4/15	
TOTAL	

NOTE:

1. **Aids Allowed:** Calculator with empty memory, to be supplied by student. Absolutely no graphing calculators allowed.
2. **Instructions:** Fill in the information on this page and ensure that the test contains 10 pages.
3. This test has 10 multiple choice questions worth 4 marks each and 4 written-answer questions worth 15 marks each.

   For the **multiple choice questions** indicate your answers by circling the appropriate letters (**A**, **B**, **C**, **D**, or **E**) on **this page (page 1)**. A multiple choice question left blank on **this page**, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the **written-answer questions**, present your solutions in the spaces provided. Use the multiple choice question pages or the back of any of the pages for rough work, for any of the questions.

ANSWER BOX FOR PART A
Circle the correct answer.

1.	A	B	C	D	E
2.	A	B	C	D	E
3.	A	B	C	D	E
4.	A	B	C	D	E
5.	A	B	C	D	E
6.	A	B	C	D	E
7.	A	B	C	D	E
8.	A	B	C	D	E
9.	A	B	C	D	E
10.	A	B	C	D	E

PART A. Multiple Choice

1. *[4 marks]*

$\lim_{x \to 2} \frac{x^4 - 16}{x - 2} =$ (call this L). There are at least 3 solution methods.

A 8 — ① Using a definition. Let $f(x) = x^4$, so $f'(x) = 4x^3$.

B 32 — $L = \lim_{x \to 2} \frac{f(x) - f(2)}{x-2} = f'(2)$ by the definition

C 16

D 4 — of derivative in HP at the top of page 484

E 64 — (12th Edition). $f'(2) = 4 \cdot 2^3$.

② Factoring. $L = \lim_{x \to 2} \frac{(x^2-4)(x^2+4)}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+2)(x^2+4)}{x-2} =$

$\lim_{x \to 2} (x+2)(x^2+4) = (2+2)(2^2+4)$

③ L'Hôpital's rule. L is indeterminate of type $\frac{0}{0}$, so

$L = \lim_{x \to 2} \frac{\frac{d}{dx}(x^4 - 16)}{\frac{d}{dx}(x-2)} = \lim_{x \to 2} 4x^3 = 4 \cdot 2^3$

2. *[4 marks]*

Let

$$f(x) = \begin{cases} x^2 - 4x & \text{if } x \le 1 \\ |x^2 - 4| & \text{if } 1 < x \le 4 \\ 2x + 4 & \text{if } x > 4 \end{cases}.$$

Then the points of discontinuity of f are:

A $x = 1$ and $x = 4$ only

B $x = 1$, $x = 2$, and $x = 4$ only

C $x = -2$ and $x = 2$ only

D $x = -2$, $x = 1$, $x = 2$, and $x = 4$ only

E $x = 1$ only

Since | | (absolute value) is continuous everywhere, $x=1$ and $x=4$ are the only possible points of discontinuity.

$\lim_{x \to 4^-} f(x) = 12 = \lim_{x \to 4^+} f(x)$ and $f(4) = |4^2 - 4| = 12$

but $f(1) = 1^2 - 4 \cdot 1 = -3$ and $\lim_{x \to 1^+} f(x) = |-3| = 3$

3. *[4 marks]*

The solutions of the inequality $\dfrac{(x-3)^2}{x(x-5)} \geq 0$ are:

A $0 < x < 3$ and $x > 5$ only

B $x \leq 0$ and $x \geq 5$ only

C $x < 0$, $x = 3$, and $x > 5$ only

D $x < 0$ and $3 \leq x < 5$ only

E $x < 0$ and $x > 5$ only

Since $(x-3)^2 \geq 0$ for all x, x is a solution iff $x-3=0$ or $x(x-5) > 0$.

$x(x-5) > 0$ iff $x < 0$ or $x > 5$

4. *[4 marks]*

Assuming that f is differentiable at x, $\displaystyle\lim_{h\to 0} \frac{[f(x+h)]^2 - [f(x)]^2}{h}$

A equals $2f'(x)$

B equals $[f'(x)]^2$

C equals 0

D equals $2f(x)f'(x)$

E does not exist

The limit is the derivative of $[f(x)]^2$.

5. *[4 marks]*

If $f(x) = \sqrt{1+\sqrt{x}}$, then $f'(9) =$

A $\frac{1}{24}$

B $\frac{1}{12}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

$$\frac{d}{dx}\left(1+x^{\frac{1}{2}}\right)^{\frac{1}{2}} = \frac{1}{2}\left(1+x^{\frac{1}{2}}\right)^{-\frac{1}{2}} \cdot \frac{1}{2}x^{-\frac{1}{2}}$$

by the chain rule.

$$f'(9) = \frac{1}{2}\left(1+9^{\frac{1}{2}}\right)^{-\frac{1}{2}} \cdot \frac{1}{2} \cdot 9^{-\frac{1}{2}}$$

$$= \frac{1}{2}(1+3)^{-\frac{1}{2}} \cdot \frac{1}{2} \cdot \frac{1}{3}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3}$$

$f(1)$

6. *[4 marks]*

The line, which is tangent at $(1, e)$ to the graph of $f(x) = x^e e^x$, has equation $y =$

A $(e^2+e)x + e^2$

B $(e^2+e)x - e^2$

C $(e^2+e)x + 1 - e^2 - e^3$

D $(e^2-e)x - e^2$

E ex

$$f'(x) = \left(ex^{e-1} + x^e\right)e^x$$

$$f'(1) = (e+1)e = e^2+e$$

Tangent line has equation

$$y = (e^2+e)(x-1) + e$$

$f(1)$

7. *[4 marks]*

If x and $y(x)$ satisfy $x^2 + xy + y^2 = e^x - e^y + 3$, then when $(x, y) = (1, 1)$, $\frac{dy}{dx} =$

A $\frac{e+3}{e-3}$

B 1

C $\frac{3+e}{3-e}$

D $\frac{3-e}{3+e}$

E $\frac{e-3}{e+3}$

Implicit differentiation:

$$2x + y + (x+2y)\frac{dy}{dx} = e^x - e^y \frac{dy}{dx}$$

When $(x,y) = (1,1)$,

$$3 + 3\frac{dy}{dx} = e - e\frac{dy}{dx}$$

$$(e+3)\frac{dy}{dx} = e - 3$$

8. *[4 marks]*

Let $f(x) = \frac{(x+1)^2}{x(x+2)}$. Then $f'(1) =$

A $\frac{4}{3}$

B $-\frac{4}{9}$

C 0

D -1

E $\frac{4}{9}$

Logarithmic differentiation:

$$\ln f = 2\ln(x+1) - \ln x - \ln(x+2)$$

$$\frac{f'}{f} = \frac{2}{x+1} - \frac{1}{x} - \frac{1}{x+2}$$

When $x=1$, $f(1) = \frac{4}{3}$ and $f'(1) = \left(\frac{2}{2} - \frac{1}{1} - \frac{1}{3}\right)\frac{4}{3}$

Alternatively, $\frac{d}{dx}\left(\frac{(x+1)^2}{x^2+2x}\right) = \frac{2(x+1)(x^2+2x) - (x+1)^2(2x+2)}{(x^2+2x)^2}$

and $f'(1) = \frac{2\cdot 2\cdot 3 - 2^2\cdot 4}{3^2}$

9. *[4 marks]*

If $x_1 \neq 1$ is used as an initial estimate for a root of $f(x) = 0$ where $f(x) = \dfrac{x}{1-x}$, then Newton's method yields the second estimate $x_2 =$

A $\sqrt{x_1}$

B 0

C $2x_1$

D $x_1{}^2$

E $\dfrac{x_1}{2}$

$$f'(x) = \frac{1}{(1-x)^2}$$

$$\text{so } x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= x_1 - \frac{\left(\frac{x_1}{1-x_1}\right)}{\left(\frac{1}{(1-x_1)^2}\right)}$$

$$= x_1 - x_1(1-x_1)$$

10. *[4 marks]*

On the interval $[-1, 4)$, the function $f(x) = \dfrac{x^3}{3} - 2x^2 + 3x + 2$ has

A a relative minimum at $x = -1$, a relative maximum at $x = 3$, and an absolute maximum at $x = 1$

B an absolute minimum at $x = 3$

C an absolute minimum at $x = -1$ and an absolute maximum at $x = 1$

D a relative maximum at $x = 1$ and no absolute minimum

E a relative minimum at $x = 3$ and no absolute maximum

$f'(x) = x^2 - 4x + 3 = (x-1)(x-3)$ so f increases on $(-1, 1)$, decreases on $(1, 3)$, and increases on $(3, 4)$.

$f(-1) = -\frac{10}{3}$, $f(1) = \frac{10}{3}$, $f(3) = 2$, and $\lim_{x \to 4^-} f(x) = \frac{10}{3}^-$.

abs. min. abs. max.

PART B. Written-Answer Questions

B1 . *[15 marks]*

In each part-question below, find the limit, or show that it does not exist.
Show your work.

B1.(a) *[5 marks]*

$$\lim_{x \to -3^-} \frac{3+x}{|9-x^2|} = \lim_{x\to -3^-} \frac{3+x}{x^2-9}$$

(if $x < -3$, $x^2 > 9$, and $9 - x^2 < 0$)

$$= \lim_{x \to -3^-} \frac{\cancel{3+x}}{\cancel{(x+3)}(x-3)} = -\frac{1}{6}$$

B1.(b) *[5 marks]*

$$\lim_{x\to-\infty} x\left(x+\sqrt{x^2-10}\right) = \lim_{x\to-\infty} \frac{x\left(x^2-(x^2-10)\right)}{x-\sqrt{x^2-10}}$$

$$= \lim_{x\to-\infty} \frac{10x}{x-\sqrt{x^2-10}}$$

$\left(\begin{array}{l} x<0; \\ x = -\sqrt{x^2} \end{array}\right) \rightarrow$

$$= \lim_{x\to-\infty} \frac{10}{1+\sqrt{1-\frac{10}{x^2}}} = 5$$

B1.(c) *[5 marks]*

$\lim_{x\to 1} \frac{ex - e^x}{x - 1 - \ln x}$ is indeterminate of type $\frac{0}{0}$, so by L'Hôpital's rule, equals $\lim_{x\to 1} \frac{e-e^x}{1-x^{-1}}$. This is again indeterminate of type $\frac{0}{0}$ so, again by L'Hôpital's rule, equals $\lim_{x\to 1} \frac{-e^x}{x^{-2}} = -e$

B2. *[15 marks]*

The demand function for a certain product is $p = 300 - q$ when $0 < q < 300$.

B2.(a) *[5 marks]*

Determine the marginal revenue when $q = 25$.

Let $r(q)$ denote total revenue;

$r(q) = q\,p(q) = 300q - q^2$

Marginal revenue $= r'(q) = 300 - 2q$

$r'(25) = 250$

B2.(b) *[5 marks]*

Find the point elasticity of demand in terms of q.

Let η denote point elasticity of demand;

$$\eta = \frac{\left(\frac{p}{q}\right)}{\left(\frac{dp}{dq}\right)} = \frac{\left(\frac{300-q}{q}\right)}{-1} = -\left(\frac{300-q}{q}\right)$$

B2.(c) *[5 marks]*

For which q is demand elastic?

Demand is elastic provided $|\eta| > 1$;

that is $\frac{300-q}{q} > 1$ iff $300 - q > q$

iff $300 > 2q$ iff $q < 150$

B3. *[15 marks]*

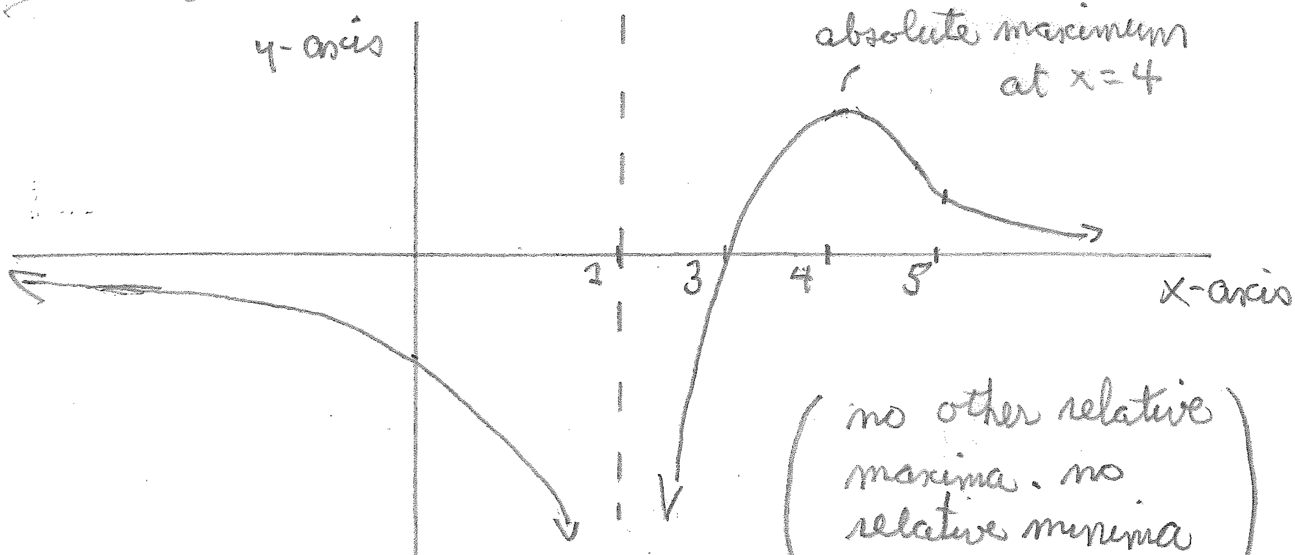
Sketch the graph of $y = \dfrac{4x-12}{(x-2)^2}$. On or near the sketch indicate any horizontal asymptotes, vertical asymptotes, critical points, intervals where y is increasing, intervals where y is decreasing, relative minima, relative maxima, absolute minima, absolute maxima, inflection points, intervals where y is concave up, and intervals where y is concave down. **Note:** $y' = \dfrac{4(4-x)}{(x-2)^3}$ and $y'' = \dfrac{8(x-5)}{(x-2)^4}$.

$\lim_{x \to \infty} \dfrac{4x-12}{(x-2)^2} = \lim_{x\to\infty} \dfrac{\frac{4}{x} - \frac{12}{x^2}}{(1-\frac{2}{x})^2} = 0$ and similarly $\lim_{x\to -\infty} y = 0$. Also $\lim_{x \to 2} y = -\infty$ (2-sided limit).

$y' < 0$ • $y' > 0$ $y'=0$ $y'<0$; 2, 4, x-axis

$y'' < 0$ • $y'' < 0$ $y''=0$ $y''>0$; 2, 5, x-axis

horizontal asymptote: $y=0$. vertical asymptote: $x=2$. critical point: $x=4$. y decreases on $(-\infty, 2)$, increases on $(2,4)$, decreases on $(4,\infty)$. inflection point: $x=5$. y is concave down on $(-\infty, 2)$ and on $(2,5)$ (but not on $(-\infty, 5)$), and is concave up on $(5, \infty)$.



(no other relative maxima. no relative minima whatsoever.)

B4. *[15 marks]*

The Eight Ball Pool Hall has 36 pool tables, which it rents by the hour. The management can rent all the tables if they charge \$5.00 per hour (per table) but each increase of \$0.25 in hourly rent causes one table to be vacant. **Show your work.**

B4.(a) *[8 marks]*

What hourly rent (per table) should the Eight Ball Pool Hall charge to maximize its hourly <u>revenue</u>?

Let x denote the number of \$0.25 increases <u>above</u> \$5.00 per hour and let $R(x)$ denote hourly revenue, depending on x.

$R(x) = (36-x)(5+.25x) = -.25x^2+4x+180$

and $R'(x) = -.5x+4$, so $x=8$ and (because of B4.(b)) <u>maximizes</u> revenue.

When $x=8$, hourly rent $= 5+.25\cdot 8 = 7$ (dollars)

B4.(b) *[2 marks]*

Show that your solution to question B4.(a) does indeed <u>maximize</u> the pool hall's hourly revenue. <u>Two</u> solutions are given here.

First derivative test: if $x<8$, $R'>0$ and if $x>8$, $R'<0$.

Second derivative test: $R'' = -.5 < 0$ for all x.

B4.(c) *[5 marks]*

The cost of operating the Eight Ball Pool Hall is \$0.50 per hour for each table rented out. What hourly rent (per table) should it charge to maximize its hourly <u>profit</u>?

Let $C(x)$ denote hourly cost and $P(x)$, hourly profit, both depending on x. $P(x) = R(x) - C(x)$

$= (-.25x^2+4x+180) - .5(36-x)$

$= -.25x^2+4.5x+162$. $P'(x) = -.5x+4.5$ so

$x=9$ is critical and the hourly <u>rent</u> which maximizes P is $5+.25\cdot 9 = 7.25$ (dollars).