MAT 133Y1Y TERM TEST 2

Thursday, 7 July, 2011, 10:30 am – 12:30 pm

Code 1

FAMILY NAME Solutions

GIVEN NAME(S) ________________

STUDENT NO. ________________

SIGNATURE ________________

GRADER'S REPORT	
Question	Mark
MC/40	
B1/15	
B2/15	
B3/15	
B4/15	
TOTAL	

NOTE:

1. **Aids Allowed:** Calculator with empty memory, to be supplied by student. Absolutely no graphing calculators allowed.
2. **Instructions:** Fill in the information on this page and ensure that the test contains 12 pages.
3. This test has 10 multiple choice questions worth 4 marks each and 4 written-answer questions worth 15 marks each.

   For the **multiple choice questions** indicate your answers by circling the appropriate letters (**A**, **B**, **C**, **D**, or **E**) on **this page (page 1)**. A multiple choice question left blank on **this page**, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the **written-answer questions**, present your solutions in the spaces provided. Use the multiple choice question pages or the back of any of the pages for rough work, for any of the questions.

ANSWER BOX FOR PART A

Circle the correct answer.

1.	**A**	**B**	**C**	**D**	**E**
2.	**A**	**B**	**C**	**D**	**E**
3.	**A**	**B**	**C**	**D**	**E**
4.	**A**	**B**	**C**	**D**	**E**
5.	**A**	**B**	**C**	**D**	**E**
6.	**A**	**B**	**C**	**D**	**E**
7.	**A**	**B**	**C**	**D**	**E**
8.	**A**	**B**	**C**	**D**	**E**
9.	**A**	**B**	**C**	**D**	**E**
10.	**A**	**B**	**C**	**D**	**E**

PART A. Multiple Choice

1. *[4 marks]*

$$\lim_{x\to 1}\left(\frac{4x}{x^2-1}-\frac{2}{x-1}\right) = \lim_{x\to 1} \frac{4x-2(x+1)}{(x-1)(x+1)}$$

$$= \lim_{x\to 1} \frac{2(x-1)}{(x-1)(x+1)}.$$

A $=-2$

B $=1$

C $=2$

D $=0$

E does not exist

2. *[4 marks]*

Let $f(x)=\begin{cases} 2^{\frac{1}{x+1}}+a & \text{if} \quad x<-1 \\ \dfrac{x^2-1}{x-1} & \text{if} \quad -1\le x<1 \\ b-x & \text{if} \quad x\ge 1 \end{cases}$.

If f is continuous at $x=-1$ and $x=1$, then

A $a=-1$ and $b=-1$

B $a=0$ and $b=1$

C $a=0$ and $b=3$

D $a=2$ and $b=3$

E $a=0$ and $b=2$

$f(-1) = \lim_{x\to -1^-} 2^{\frac{1}{x+1}} + a = a$

because as $x\to -1^-$, $x+1\to 0^-$, so $\frac{1}{x+1}\to -\infty$ and $2^{\frac{1}{x+1}}\to 0$.

Also $f(-1) = \frac{(-1)^2-1}{(-1)-1} = 0$

Similarly, $f(1) = \lim_{x\to 1^-} \frac{x^2-1}{x-1} = 2$ and $f(1) = b-1$

3. *[4 marks]*

The equation of the line tangent to the graph of $y = \dfrac{(x^2-3)^3}{\sqrt{4-3x}}$ at the point $(1, -8)$ is

A $y = 16x - 24$

B $y = 12x - 20$

C $2y = 51x - 67$

D $y = -12x + 4$

E $y = -16x + 8$

$\ln y = 3\ln(x^2-3) - \frac{1}{2}\ln(4-3x)$

so that

$\dfrac{y'}{y} = \dfrac{6x}{x^2-3} + \dfrac{3}{2(4-3x)}$

when $x = 1$, $y = -8$, and $\dfrac{y'}{-8} = \dfrac{6}{-2} + \dfrac{3}{2} = -\dfrac{3}{2}$

So $y' = 12$ and the line has equation

$y = 12(x-1) - 8$

4. *[4 marks]*

If $y^3 + xy = 2x^2$, then when $x = 1$ and $y = 1$, $\dfrac{dy}{dx} =$

A $\frac{3}{4}$

B 1

C $\frac{1}{2}$

D $\frac{5}{4}$

E $\frac{1}{4}$

Implicit differentiation:

$3y^2\dfrac{dy}{dx} + x\dfrac{dy}{dx} + y = 4x$

If $x = 1, y = 1$,

$3\dfrac{dy}{dx} + \dfrac{dy}{dx} + 1 = 4$

$f(4) = 4^{\sqrt{4}} = 16$

5. *[4 marks]*

If $f(x) = x^{\sqrt{x}}$, then $f'(4) =$

A $8 + 8\ln 4$

B $4 + 4\ln 4$

C $4 + 8\ln 4$

D $8 + 4\ln 4$

E $4 - 8\ln 4$

$\ln f(x) = \sqrt{x} \ln x$

$$\frac{f'(x)}{f(x)} = \frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}$$

$$\frac{f'(4)}{16} = \frac{1}{\sqrt{4}} + \frac{\ln 4}{2\sqrt{4}}$$

$$= \frac{1}{2} + \frac{\ln 4}{4}$$

6. *[4 marks]*

If a product has demand function $p = e^{-2q}$, then its point elasticity of demand when $q = 4$ is

A $-\frac{1}{4}$

B $-\frac{1}{8}$

C $-\frac{1}{16}$

D $-\frac{1}{2}$

E -1

$\frac{dp}{dq} = -2e^{-2q}$ so that

$$\frac{\left(\frac{p}{q}\right)}{\left(\frac{dp}{dq}\right)} = \frac{\left(\frac{e^{-2q}}{q}\right)}{-2e^{-2q}} = -\frac{1}{2q}$$

7. *[4 marks]*

If Newton's method is used to approximate a root of the equation $x^4 - 4x + 1 = 0$ by taking $x_1 = 0$ as the first estimate, then the third estimate (x_3) will be closest to

A 0.25079

B 0.25000

C 0.25039

D 0.25099

E 0.24784

Let $f(x) = x^4 - 4x + 1$

$f'(x) = 4x^3 - 4$

$x_2 = 0 - \dfrac{f(0)}{f'(0)} = \dfrac{1}{4}$

$x_3 = \dfrac{1}{4} - \dfrac{f(\frac{1}{4})}{f'(\frac{1}{4})} = \dfrac{1}{4} - \dfrac{\left(\frac{1}{256}\right)}{\left(\frac{1}{16} - 4\right)}$

$= \dfrac{252}{1008} - \dfrac{1}{16 - 1024} = \dfrac{253}{1008}$

8. *[4 marks]*

$$\lim_{x \to 1^+} \frac{x^{\frac{1}{2}} - 1}{e^x - ex} =$$

A $\dfrac{1}{2e}$

B $\dfrac{2}{e}$

C $2e$

D $\dfrac{e}{2}$

E ∞

As $x \to 1^+$, $x^{\frac{1}{2}} - 1 \to 0$ and $e^x - ex \to 0$ so the limit is indeterminate of type $\frac{0}{0}$ and equals $\lim\limits_{x \to 1^+} \dfrac{\frac{1}{2}x^{-\frac{1}{2}}}{e^x - e}$ by L'Hôpital.

As $x \to 1^+$, $\frac{1}{2}x^{-\frac{1}{2}} \to \frac{1}{2}$ and $e^x - e \to 0^+$ so $\lim\limits_{x \to 1^+} \dfrac{\frac{1}{2}x^{-\frac{1}{2}}}{e^x - e}$ is not indeterminate and is (positive) infinite.

9. *[4 marks]*

$\lim_{x\to\infty}(1+\frac{1}{3}x^{-1})^{7+2x} =$ (call the limit y)

A 0

B 1

C e^7

D $e^{\frac{2}{3}}$

E 7

$$\ln y = \lim_{x\to\infty} \frac{\ln(1+\frac{1}{3}x^{-1})}{(7+2x)^{-1}} \quad (\text{type } \tfrac{0}{0})$$

$$= \lim_{x\to\infty} \frac{(1+\frac{1}{3}x^{-1})^{-1}\cdot(-\frac{1}{3}x^{-2})}{-(7+2x)^{-2}\cdot 2}$$

$$= \lim_{x\to\infty} (1+\tfrac{1}{3}x^{-1})^{-1}\frac{(7+2x)^2}{6x^2} = \lim_{x\to\infty}(1+\tfrac{1}{3}x^{-1})^{-1}\cdot\tfrac{1}{6}\cdot(\tfrac{7}{x}+2)^2$$

$= 1\cdot\frac{1}{6}\cdot 2^2$. With $\ln y = \frac{2}{3}$, $y = e^{\frac{2}{3}}$.

10. *[4 marks]*

The function $f(x) = \dfrac{x}{(x-1)^2}$ on the interval $[-2, 2]$ has

A an absolute minimum at $x = 2$ and an absolute maximum at $x = -2$

B an absolute minimum at $x = -2$ and no absolute maximum

C an absolute minimum at $x = -1$ and no absolute maximum

D an absolute minimum at $x = -1$ and an absolute maximum at $x = 2$

E no absolute minimum or maximum

$\lim_{x\to 1} f(x) = +\infty$ so f has no absolute maximum

$$f'(x) = \frac{(x-1)^2 - 2x(x-1)}{(x-1)^4} = \frac{1-x^2}{(x-1)^4}.$$

f decreases on $(-2,-1)$, increases on $(-1,1)$, and decreases on $(1,2)$ — and is continuous at -2 and 2. $f(-1) = -\frac{1}{4}$ and $f(2) = 2$; $-\frac{1}{4} < 2$.

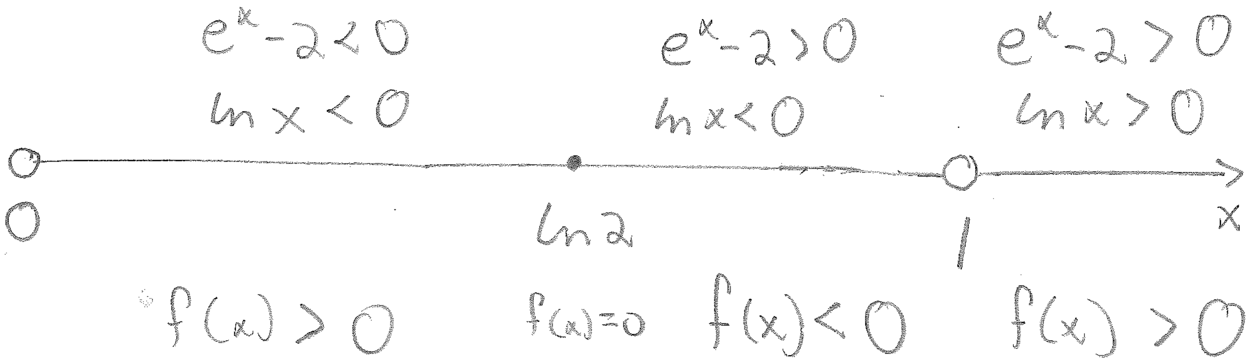
PART B. Written-Answer Questions
SHOW YOUR WORK.

B1. *[15 marks]*

Solve the inequality $\dfrac{e^x - 2}{\ln x} > 0$.

Let $f(x) = \dfrac{e^x - 2}{\ln x}$ for $x > 0$ and $x \neq 1$

Sign of $f(x)$:



Solutions of the inequality are in $(0, \ln 2)$ and $(1, \infty)$:

$0 < x < \ln 2$ or $x > 1$

B2.(a) *[8 marks]*

If a demand function is given by $p = \dfrac{100 + 500q^{-\frac{1}{2}}}{175 + q^2}$, find the marginal revenue when $q = 25$.

Let $r(q) = q \cdot p(q)$ (revenue)

$$r(q) = \frac{100q + 500q^{\frac{1}{2}}}{175 + q^2}$$

$$r'(q) = \frac{(100 + 250q^{-\frac{1}{2}})(175 + q^2) - (100q + 500q^{\frac{1}{2}})(2q)}{(175 + q^2)^2}$$

$$r'(25) = \frac{150 \cdot 800 - 5000 \cdot 50}{(800)^2}$$

$$= \frac{12 - 25}{64}$$

$$= -\frac{13}{64}$$

B2.(b) *[7 marks]*

A refrigerator company finds that the average cost per refrigerator to produce q refrigerators is given in dollars by the function $\bar{c}(q) = 0.01q^2 - q + 70 + \dfrac{3000}{q}$.

(i) *[1 mark]* What is the total cost of producing q refrigerators?

(ii) *[3 marks]* What is the marginal cost if 10 refrigerators are made?

(iii) *[3 marks]* What is the relative rate of change of cost with respect to the number of refrigerators made when 10 refrigerators are made?

(i) Let $c(q)$ = total cost

$$c(q) = q\bar{c}(q) = 0.01q^3 - q^2 + 70q + 3000$$

(ii) $c'(q) = 0.03q^2 - 2q + 70$

$$c'(10) = 3 - 20 + 70 = 53$$

(iii) $c(10) = 10 - 100 + 700 + 3000$

$$= 3610$$

The relative rate of change of c when $q = 10$ is $\dfrac{c'(10)}{c(10)} = \dfrac{53}{3610}$

B3. *[15 marks]*

Let $f(x) = xe^{-x}$.

B3.(a) *[10 marks]*

Find the following:

(i) *[2 marks]* all horizontal and vertical asymptotes (if any — justify your answer)

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x}$ by L'Hôpital's rule

But $\lim_{x\to\infty} \frac{1}{e^x} = 0$ so $y=0$ is the horizontal asymptote. (No vertical asymptote; also $f(x)\to-\infty$ as $x\to-\infty$, so no other horiz. asymptote)

(ii) *[4 marks]* all intervals where f is increasing (if any), all intervals where f is decreasing (if any), all local minima (if any), and all local maxima (if any)

$f'(x) = (1-x)e^{-x}$

f increases on $(-\infty, 1)$, decreases on $(1, \infty)$

and takes a local maximum at $x=1$.

(no minimum)

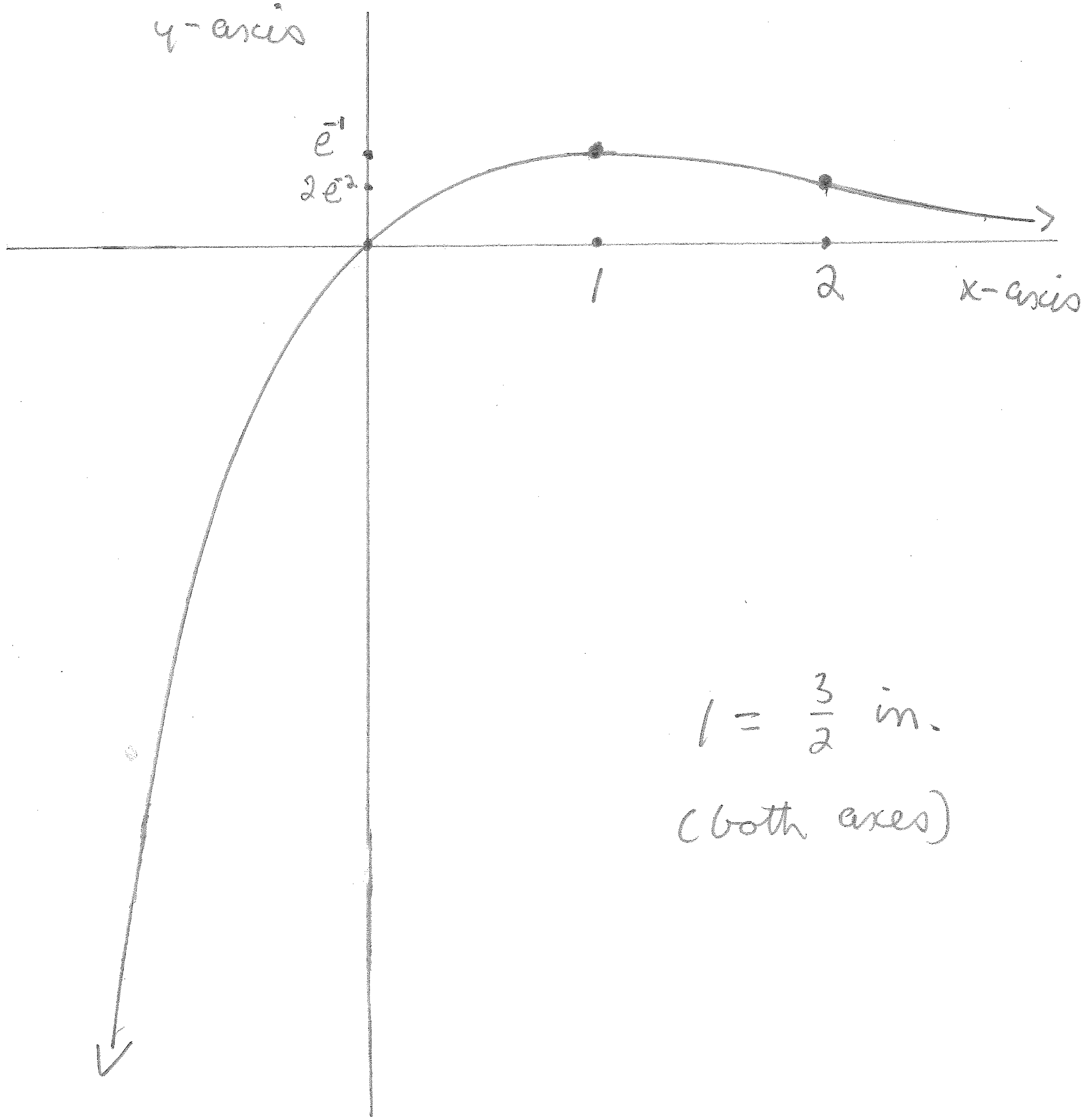
(iii) *[4 marks]* all intervals where f is concave up (if any), all intervals where f is concave down (if any), and all inflection points (if any)

$f''(x) = (x-2)e^{-x}$

f is concave down on $(-\infty, 2)$, concave up on $(2, \infty)$, and has inflection point $x=2$.

B3.(b) *[5 marks]*

Sketch the graph of $f(x) = xe^{-x}$.



B4. *[15 marks]*

If a university charges \$12 for a football ticket, it sells on average 70,000 tickets. For every \$1 increase in the ticket price it loses 2000 in attendance. If every ticket buyer spends \$3 on refreshments, what price per ticket should the university charge to maximize its revenue? **Show your work**. Remember to **verify** that your answer indeed maximizes revenue.

Let $x =$ the number of dollars charged per ticket, in excess of \$12, and let $f(x)$ denote revenue for $x > 0$.

Then the revenue earned per ticket sold is $x + 12 + 3 = x + 15$ dollars and $70{,}000 - 2{,}000x$ tickets are sold;

$f(x) = (x+15)(70{,}000 - 2000x)$ dollars

$= -2{,}000x^2 + 40{,}000x + 1{,}050{,}000$

To maximize f, $f'(x) = -4{,}000x + 40{,}000$

so $x = 10$ is critical.

Since $f''(x) = -4{,}000 < 0$ for all x in $(0, \infty)$,

$x = 10$ maximizes revenue.

The university should charge

$10 + 12 =$ 22 dollars per ticket.