MAT 133Y1Y TERM TEST 2

Thursday, 5 July, 2012, 10:30 am – 12:30 pm

Code 1

FAMILY NAME Solutions

GIVEN NAME(S) ____________________

STUDENT NO. ____________________

SIGNATURE ____________________

GRADER'S REPORT	
Question	Mark
MC/40	
B1/15	
B2/15	
B3/15	
B4/15	
TOTAL	

NOTE:

1. **Aids Allowed:** Calculator with empty memory, to be supplied by student. Absolutely no graphing calculators allowed.
2. **Instructions:** Fill in the information on this page and ensure that the test contains 11 pages.
3. This test has 10 multiple choice questions worth 4 marks each and 4 written-answer questions worth 15 marks each.

   For the **multiple choice questions** indicate your answers by circling the appropriate letters (**A**, **B**, **C**, **D**, or **E**) on **this page (page 1)**. A multiple choice question left blank on **this page**, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the **written-answer questions**, present your solutions in the spaces provided. Use the multiple choice question pages or the back of any of the pages for rough work, for any of the questions.

ANSWER BOX FOR PART A

Circle the correct answer.

1.	A	B	C	D	E
2.	A	B	C	D	E
3.	A	B	C	D	E
4.	A	B	C	D	E
5.	A	B	C	D	E
6.	A	B	C	D	E
7.	A	B	C	D	E
8.	A	B	C	D	E
9.	A	B	C	D	E
10.	A	B	C	D	E

PART A. Multiple Choice

1. *[4 marks]*

$$\lim_{x \to -2^+} \frac{4x^2}{x^2-4} =$$

A 4

B $-\infty$

C 1

D 0

E $+\infty$

If $-2 < x < 0$, $x^2 < 4$, and $x^2 - 4 < 0$.

As $x \to -2^+$, $x^2 - 4 \to 0^-$ and $4x^2 \to 16 > 0$.

(Therefore, this limit is not indeterminate and L'Hôpital's rule does not apply.)

2. *[4 marks]*

$$\lim_{x \to -1} \frac{3x^{\frac{1}{3}} + 2 - x}{(x+1)^2} = \lim_{x \to -1} \frac{x^{-\frac{2}{3}} - 1}{2(x+1)} = \lim_{x \to -1} \frac{-\frac{2}{3} x^{-\frac{5}{3}}}{2} = \frac{\left(-\frac{2}{3}\right)(-1)}{2}$$

A $\frac{1}{3}$

B 1

C $-\frac{1}{3}$

D -3

E 3

both indeterminate of type $\frac{0}{0}$

3. *[4 marks]*

$\lim_{x \to 1} x^{\left(\frac{1}{x^2-1}\right)} =$

A $\frac{1}{2}$

B e

C e^2

D 1

E $\sqrt{e}$

Let $y = \lim_{x \to 1} x^{\left(\frac{1}{x^2-1}\right)}$

Then $\ln y = \lim_{x \to 1} \frac{\ln x}{x^2-1}$

indeterminate, type $\frac{0}{0}$

$= \lim_{x \to 1} \frac{x^{-1}}{2x}$

$= \frac{1}{2}$

4. *[4 marks]*

Let $f(x) = \begin{cases} \dfrac{x+1}{x^2-1} & \text{if} \quad x < -1 \\ ax+b & \text{if} \quad -1 \leq x \leq 1 \\ e^{\left(\frac{1}{1-x}\right)} & \text{if} \quad x > 1 \end{cases}$.

If f is continuous everywhere (at all real x) then

A $a = -\frac{1}{4}$ and $b = \frac{1}{4}$

B $a = \frac{1}{2}$ and $b = -\frac{1}{2}$

C $a = \frac{1}{4}$ and $b = -\frac{1}{4}$

D $a = 0$ and $b = 0$

E $a = -\frac{1}{2}$ and $b = \frac{1}{2}$

$a(-1)+b = f(-1) = \lim_{x \to -1^-} f(x)$

$= \lim_{x \to -1^-} \frac{1}{x-1} = -\frac{1}{2}$

$a \cdot 1 + b = f(1) = \lim_{x \to 1^+} f(x) = 0$

because as $x \to 1^+$, $1-x \to 0^-$, $\frac{1}{1-x} \to -\infty$, and $e^{\frac{1}{1-x}} \to 0$.

So $-a+b = -\frac{1}{2}$, $a+b = 0$.

5. *[4 marks]*

If $f(x) = \dfrac{(2x+1)^3}{x^2+1}$, then $f'(1) =$

A 0

B 27

C 9

D $\frac{27}{2}$

E 18

$$f'(x) = \frac{3(2x+1)^2 \cdot 2(x^2+1) - (2x+1)^3 \cdot 2x}{(x^2+1)^2}$$

$$f'(1) = \frac{3 \cdot 3^2 \cdot 2 \cdot 2 - 3^3 \cdot 2 \cdot 1}{2^2}$$

Alternative method (Logarithmic differentiation), noting that $f(1) = \frac{3^3}{2} = \frac{27}{2}$:

$\ln f = 3\ln(2x+1) - \ln(x^2+1)$, so $\dfrac{f'(x)}{f(x)} = \dfrac{3 \cdot 2}{2x+1} - \dfrac{2x}{x^2+1} = 1$ when $x=1$

6. *[4 marks]*

If $y = ax + b$ is the equation of the line which is tangent at $x = 1$ to the graph of $f(x) = xe^{x-1}$, then $b =$

A -2

B 2

C -1

D 0

E 1

$f'(x) = e^{x-1} + xe^{x-1}$

$f(1) = 1$ and $f'(1) = 2$

Tangent line has equation

$y = 2(x-1) + 1 = 2x - 1$

7. *[4 marks]*

If $x^{2y} = y$, then when $x = 1$ and $y = 1$, $\dfrac{dy}{dx} =$

A 2

B 0

C 1

D 4

E $\frac{1}{2}$

$2y \ln x = \ln y$

Differentiating this implicitly,

$$2\frac{y}{x} + 2\ln x \frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

When $x = 1$ and $y = 1$, $\ln x = 0$ and $2 = \dfrac{dy}{dx}$

8. *[4 marks]*

If a product has demand function $p = e^{-2q}$, then its point elasticity of demand when $q = 4$ is

A $-\frac{1}{4}$

B $-\frac{1}{8}$

C $-\frac{1}{16}$

D $-\frac{1}{2}$

E -1

$$\frac{dp}{dq} = -2e^{-2q}$$

When $q = 4$,

$$\frac{\left(\frac{p}{q}\right)}{\left(\frac{dp}{dq}\right)} = \frac{\left(\frac{e^{-2q}}{q}\right)}{-2e^{-2q}} = -\frac{1}{2q} = -\frac{1}{8}$$

9. *[4 marks]*

If $x_1 \neq 1$ is used as an initial estimate for a solution of $f(x) = 0$, when $f(x) = \dfrac{x}{1-x}$, then Newton's method yields the second estimate $x_2 =$

A $\sqrt{x_1}$

B 0

C $2x_1$

D $x_1{}^2$

E $\frac{1}{2}x_1$

$$f'(x) = \frac{1\cdot(1-x) - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{\left(\frac{x_1}{1-x_1}\right)}{\left(\frac{1}{(1-x_1)^2}\right)}$$

$$= x_1 - x_1(1-x_1)$$

10. *[4 marks]*

Let M denote the absolute maximum value of $f(x) = 2x^3 - 3x^2 - 12x + 27$ on $[-3, 3]$ and let m denote the absolute minimum value of the same function on the same interval. Then $M + m =$

A 41

B 25

C −11

D 0

E 16

$$f'(x) = 6x^2 - 6x - 12$$
$$= 6(x+1)(x-2)$$

Critical points are $x = -3, -1, 2,$ and 3, where f takes the values $-18, 34, 7,$ and 18

least

greatest

PART B. Written-Answer Questions
SHOW YOUR WORK.

B1. *[15 marks]*

The demand equation for a certain product is $p = 50 - \sqrt{q}$.

B1.(a) *[5 marks]*

Find the marginal revenue when $q = 100$. Let r = revenue

$r = q \cdot p = 50q - q^{\frac{3}{2}}$. $r' = 50 - \frac{3}{2} q^{\frac{1}{2}}$

$r'(100) = 50 - \frac{3}{2}(100)^{\frac{1}{2}} = 35$

B1.(b) *[5 marks]*

Find the percentage rate of change of revenue with respect to q when $q = 100$.

$$\frac{r'(100)}{r(100)} = \frac{35}{5000-1000} = \frac{7}{800} = .875\%$$

B1.(c) *[5 marks]*

If m employees can produce $q = 5(m - \sqrt{m})$ units of product, find the marginal revenue product when the number of employees is 25.

$\frac{dq}{dm} = 5 - \frac{5}{2} m^{-\frac{1}{2}}$ and when $m = 25$, $\frac{dq}{dm} = \frac{9}{2}$

$$\frac{dr}{dm}(25) = \frac{dr}{dq}(q(25)) \cdot \frac{dq}{dm}(25) = \frac{dr}{dq}(100) \cdot \frac{dq}{dm}(25)$$

$$= 35 \cdot \frac{9}{2} = \frac{315}{2} = 157.5$$

B2. *[15 marks]*

Suppose $y(x)$ satisfies $y^3 + xy = 2x^2$.

B2.(a) *[7 marks]*

Find $\dfrac{dy}{dx}$ in terms of x and y.

$$3y^2 \frac{dy}{dx} + x\frac{dy}{dx} + y = 4x$$

$$(x + 3y^2)\frac{dy}{dx} = 4x - y \quad \bigstar$$

$$\frac{dy}{dx} = \frac{4x-y}{x+3y^2}$$

B2.(b) *[8 marks]*

Find $\dfrac{d^2y}{dx^2}$ when $x = 1$ and $y = 1$. $\left(\text{So that } \dfrac{dy}{dx} = \dfrac{4-1}{1+3} = \dfrac{3}{4}.\right)$

Differentiating $\bigstar$:

$$(x+3y^2)\frac{d^2y}{dx^2} + \left(1 + 6y\frac{dy}{dx}\right)\frac{dy}{dx} = 4 - \frac{dy}{dx}$$

When $x=1$ and $y=1$,

$$4\frac{d^2y}{dx^2} + \left(1 + 6\cdot\frac{3}{4}\right)\frac{3}{4} = 4 - \frac{3}{4}, \quad 4\frac{d^2y}{dx^2} = -\frac{7}{8}$$

$$\frac{d^2y}{dx^2} = -\frac{7}{32}$$

B3. *[15 marks]*

Let $f(x) = \dfrac{x^2 + x - 2}{(x+1)^2}$.

B3.(a) *[2 marks]*

Find all horizontal and vertical asymptotes of f.

Horizontal asymptote is $y = 1$ $\left(\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \dfrac{1 + \frac{1}{x} - \frac{2}{x^2}}{\left(1 + \frac{1}{x}\right)^2} \right)$

Vertical asymptote is $x = -1$ $\left(\lim_{x \to -1} f(x) = -\infty \right)$

B3.(b) *[4 marks]*

Find where f is increasing, where f is decreasing, all x where f takes a relative maximum, and all x where f takes a relative minimum. Note that $f'(x) = \dfrac{x+5}{(x+1)^3}$.

f increases on $(-\infty, -5)$ and on $(-1, \infty)$,
decreases on $(-5, -1)$.

f has a relative maximum at $x = -5$.

B3.(c) *[4 marks]*

Find where f is concave upward, where f is concave downward, and all inflection points of f. Note that $f''(x) = -\dfrac{2(x+7)}{(x+1)^4}$.

f is concave up on $(-\infty, -7)$ and

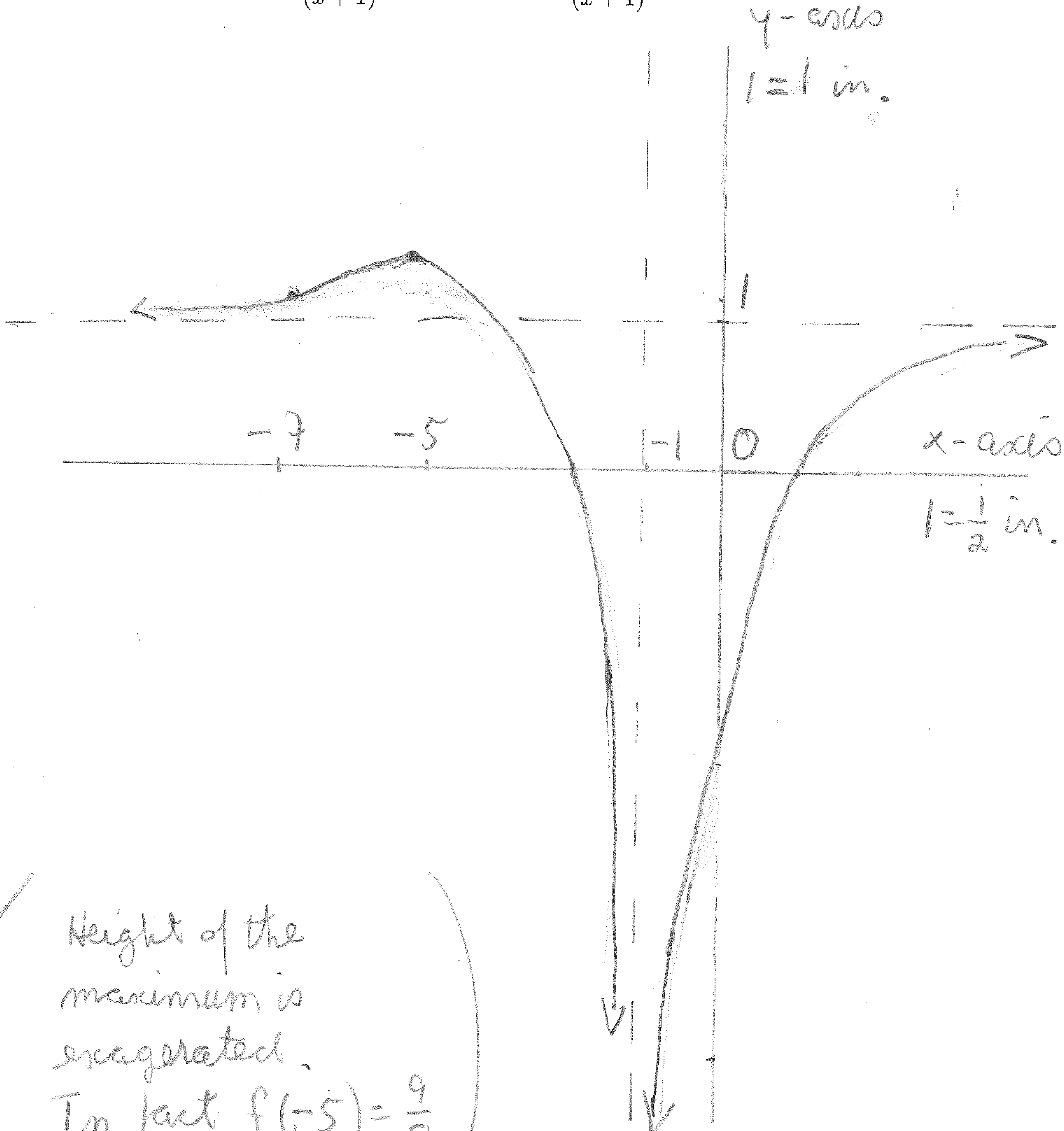
concave down on $(-7, -1)$ and $(-1, \infty$

inflection point at $x = -7$

B3.(d) *[5 marks]*

Sketch the graph of $f(x) = \dfrac{x^2 + x - 2}{(x+1)^2}$.

Recall that $f'(x) = \dfrac{x+5}{(x+1)^3}$ and $f''(x) = -\dfrac{2(x+7)}{(x+1)^4}$.



Height of the maximum is exagerated. In fact $f(-5) = \frac{9}{8}$

B4. *[15 marks]*

A landlord has 50 apartments which can all be rented if he charges \$1000 per apartment per month. For each \$25 increase in monthly rent per apartment, he will have one vacancy that he cannot fill. What monthly rent should he charge per apartment in order to maximize his monthly revenue? Remember to verify that your answer indeed maximizes revenue.

Let x = the number of \$25 increases (above \$1000) in monthly rent per apartment.

Then $50-x$ apartments are rented, each for $1000+25x$ dollars per month. Let $f(x)=(50-x)(1000+25x)$

$= 50{,}000 + 250x - 25x^2$

(total monthly revenue)

To maximize f, $f'(x) = 250-50x$

and $x=5$ is critical.

To verify that $x=5$ maximizes f, note that $f''(x) = -50 < 0$.

The landlord should charge

$1000+25\cdot 5 = 1150$ dollars per apartment (per month)