MAT 133Y1Y TERM TEST 3

Thursday, 29 July, 2010, 10:30 am – 12:30 pm

Code 1

FAMILY NAME Solutions

GIVEN NAME(S) ____________________

STUDENT NO. ____________________

SIGNATURE ____________________

GRADER'S REPORT	
Question	Mark
MC/40	
B1/15	
B2/15	
B3/15	
B4/15	
TOTAL	

NOTE:

1. **Aids Allowed:** Calculator with empty memory, to be supplied by student. Absolutely no graphing calculators allowed.
2. **Instructions:** Fill in the information on this page and ensure that the test contains 10 pages.
3. This test has 10 multiple choice questions worth 4 marks each and 4 written-answer questions worth 15 marks each.

   For the **multiple choice questions** indicate your answers by circling the appropriate letters (**A**, **B**, **C**, **D**, or **E**) on **this page (page 1)**. A multiple choice question left blank on **this page**, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the **written-answer questions**, present your solutions in the spaces provided. Use the multiple choice question pages or the back of any of the pages for rough work, for any of the questions.

ANSWER BOX FOR PART A					
Circle the correct answer.					
1.	A	B	C	D	E
2.	A	B	C	D	E
3.	A	B	C	D	E
4.	A	B	C	D	E
5.	A	B	C	D	E
6.	A	B	C	D	E
7.	A	B	C	D	E
8.	A	B	C	D	E
9.	A	B	C	D	E
10.	A	B	C	D	E

PART A. Multiple Choice

1. *[4 marks]*

If $f''(x) = 24x$, $f(0) = 4$, and $f'(0) = 6$, then $f(1) =$

A 16

B 17

C 14

D 15

E 18

$f'(x) = \int f''(x)\,dx = \int 24x\,dx = 12x^2 + C_1$

where $f'(0) = 6$ implies $6 = 12 \cdot 0^2 + C_1$

and $C_1 = 6$, $f'(x) = 12x^2 + 6$.

$f(x) = \int f'(x)\,dx = \int 12x^2 + 6\,dx = 4x^3 + 6x + C_0$

where $f(0) = 4$ implies $4 = 4 \cdot 0^3 + 6 \cdot 0 + C_0$

and $C_0 = 4$, $f(x) = 4x^3 + 6x + 4$,

$f(1) = 4 \cdot 1^3 + 6 \cdot 1 + 4$.

2. *[4 marks]*

If $f(x) = \int_0^x \frac{dt}{1+e^t}$, then $f'(\ln 2) =$

A $\frac{1}{3}$

B $\ln\left(\frac{2}{3}\right)$

C $-\frac{2}{9}$

D $-\ln 3$

E $\ln\left(\frac{4}{3}\right)$

By the fundamental theorem of calculus, $f'(x) = \frac{1}{1+e^x}$

Thus $f'(\ln 2) = \frac{1}{1+e^{(\ln 2)}}$

PART A. Multiple Choice

3. *[4 marks]*

$\int_0^1 x^3(1+x^4)^5 \, dx =$ $\int_1^2 u^5 \frac{du}{4} = \frac{1}{24}\left[u^6\right]_1^2 = \frac{2^6-1^6}{24}$

A $\frac{21}{2}$

B $\frac{21}{8}$

C $\frac{1}{24}$

D $\frac{1}{4}$

E $\frac{13}{8}$

By substitution, where

$u = 1 + x^4$,

$du = 4x^3 dx$, so $\frac{du}{4} = x^3 dx$, and

$u = 1$ when $x = 0$, $u = 2$ when $x = 1$.

4. *[4 marks]*

If $f(x) = \begin{cases} x & \text{if } x < 2 \\ 2 & \text{if } x \geq 2 \end{cases}$ then $\int_{-1}^{6} f(x) \, dx =$ $\int_{-1}^{2} x \, dx + \int_{2}^{6} 2 \, dx$

A 10

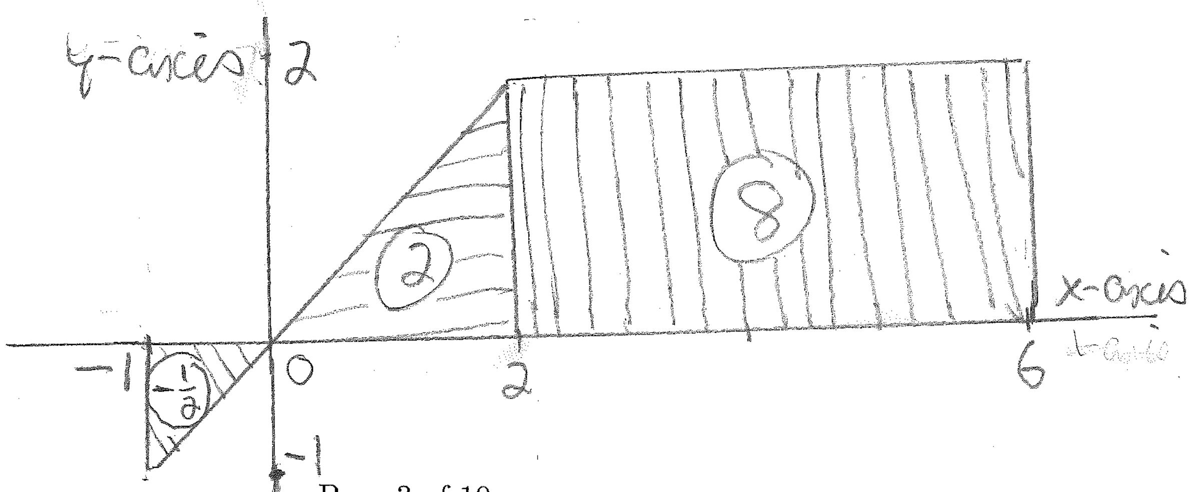
B $11\frac{1}{2}$

C $9\frac{1}{2}$

D 11

E 9

$= \left[\frac{x^2}{2}\right]_{-1}^{2} + \left[2x\right]_{2}^{6} = 1\frac{1}{2} + 8$



5. *[4 marks]*

The area of the region bounded by the x-axis, the line $x = 2$ and the curve $y = x^{\frac{1}{3}}$ is

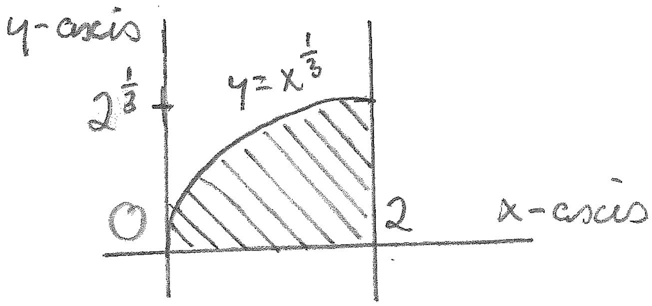
A $\frac{8}{3}\sqrt[3]{2}$

B $\frac{3}{2}\sqrt[3]{2}$

C $1 - \sqrt[3]{2}$

D 2

E $8\sqrt[3]{2} - 1$



$$\int_0^2 x^{\frac{1}{3}}\,dx = \frac{3}{4}\left[x^{\frac{4}{3}}\right]_0^2$$

$$= \frac{3}{4}\cdot 2^{\frac{4}{3}} = \frac{3}{4}\cdot 2 \cdot 2^{\frac{1}{3}}$$

6. *[4 marks]*

A manufacturer's marginal cost function is

$$\frac{dc}{dq} = \frac{500}{\sqrt{2q+40}}$$

where c is in dollars. The cost to increase production from 12 to 30 units is

A \$2000

B \$500

C \$800

D \$1000

E \$100

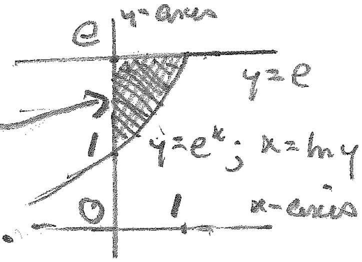
$$\int_{12}^{30} 500\,(2q+40)^{-\frac{1}{2}}\,dq$$

$$= 500\left[(2q+40)^{\frac{1}{2}}\right]_{12}^{30}$$

$$= 500\left[100^{\frac{1}{2}} - 64^{\frac{1}{2}}\right]$$

7. *[4 marks]*

The area bounded by the y-axis, the straight line $y = ex$ and the curve $y = e^x$ is equal to



A $\displaystyle\int_0^1 e - e^x \, dx$

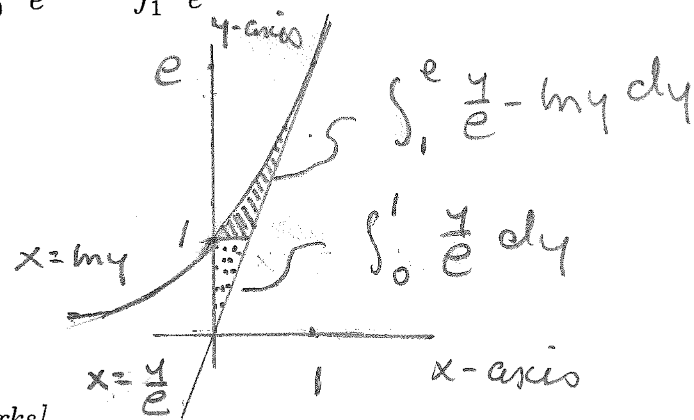
B $\displaystyle\int_1^e \ln y \, dy$

No. These are equal to each other; Both equal this area.

C $\displaystyle\int_0^e e^x - ex \, dx$ No. The area is actually $\int_0^1 e^x - ex \, dx$.

D $\displaystyle\int_1^e \ln y - \frac{y}{e} \, dy$ No. This is the negative of the area shown here.

E $\displaystyle\int_0^1 \frac{y}{e} \, dy + \int_1^e \frac{y}{e} - \ln y \, dy$



8. *[4 marks]*

The average value of $f(x) = \sqrt{x-4}$ over the interval $[5, 13]$ is

A $\frac{13}{4}$

B $\frac{13}{6}$

C $\frac{52}{3}$

D $\frac{1}{6}$

E 2

$$\frac{1}{13-5} \int_5^{13} (x-4)^{\frac{1}{2}} \, dx$$

$$= \frac{1}{12} \left[(x-4)^{\frac{3}{2}} \right]_5^{13}$$

$$= \frac{1}{12} \left[9^{\frac{3}{2}} - 1^{\frac{3}{2}} \right]$$

9. *[4 marks]*

$$\int_0^\infty \frac{1}{(x+2)^5}\,dx = \lim_{b\to\infty} \int_0^b (x+2)^{-5}\,dx$$

$$= \lim_{b\to\infty} -\frac{1}{4}\left[(x+2)^{-4}\right]_0^b$$

$$= \lim_{b\to\infty} \frac{1}{4}\left[(x+2)^{-4}\right]_b^0$$

$$= \lim_{b\to\infty} \frac{1}{4}\left[2^{-4} - (b+2)^{-4}\right]$$

$$= \frac{1}{4}\cdot 2^{-4} \text{ because as } b\to\infty,\ (b+2)^{-4}\to 0$$

A diverges

B $=0$

C $=\frac{1}{16}$

D $=-\frac{1}{16}$

E $=\frac{1}{64}$

10. *[4 marks]*

If $\frac{dy}{dx} = -xy$ and $y=2$ when $x=0$ then when $x=2$, $y=$

A 0

B 1

C e^{-4}

D $2e^{-2}$

E e^{-2}

Separate variables:

$$\frac{dy}{y} = -x\,dx$$

Integrate both sides:

$$\ln|y| = C - \frac{x^2}{2} \text{ (general solution)}$$

when $x=0, y=2$: $\ln|2| = C - \frac{0^2}{2}$, so $C=\ln 2$

and $\ln|y| = (\ln 2) - \frac{x^2}{2}$ (particular solution).

That is, $|y(x)| = 2e^{-\frac{x^2}{2}}$. Thus $|y(2)| = 2e^{-2}$. We did not determine the <u>sign</u> of $y(2)$, but only D can be true.

PART B. Written-Answer Questions
SHOW YOUR WORK.

B1. *[15 marks]*

Find the area of the region bounded by the curves $y = 5\ln x$ and $y = x\ln x$.
[A rough sketch may be helpful, but is **not** required.]

If (x,y) is a point of intersection, then $5\ln x = x\ln x$, $(5-x)\ln x = 0$, and $x=5$ or $\ln x = 0$ (which implies $x=1$).

When x ranges in $[1,5]$, $(5-x)\ln x \geq 0$ and the curve $y = 5\ln x$ lies above the curve $y = x\ln x$.

Thus the area is $\int_1^5 (5-x)\ln x\, dx$.

Integrating by parts with $u = \ln x$, $dv = (5-x)\,dx$, and $du = \frac{dx}{x}$, $v = 5x - \frac{1}{2}x^2$ yields $\int (5-x)\ln x\, dx = \left(5x - \frac{1}{2}x^2\right)\ln x - \int\left(5x - \frac{1}{2}x^2\right)\frac{dx}{x}$

$= \left(5x - \frac{1}{2}x^2\right)\ln x - 5x + \frac{1}{4}x^2$.

The evaluation of the area is

$\left[\left(5x - \frac{1}{2}x^2\right)\ln x - 5x + \frac{1}{4}x^2\right]_1^5 = \frac{25}{2}\ln 5 - \frac{75}{4} - \left(-\frac{19}{4}\right)$

$= \frac{25}{2}\ln 5 - 14$

B2. *[15 marks]*

Find $\int \frac{x^3}{(1+x^2)^2}\,dx$. [Suggestion: $u = 1+x^2$, but there are other ways too; partial fractions is **not** one of these ways.]

With $u = 1+x^2$, $du = 2x\,dx$, so

$x^2 = u-1$ and $x\,dx = \frac{du}{2}$.

Then $\int \frac{x^3}{(1+x^2)^2}\,dx = \int \frac{x^2}{(1+x^2)^2}\,x\,dx$

$= \int \frac{u-1}{u^2}\,\frac{du}{2} = \int \frac{1}{2}u^{-1} - \frac{1}{2}u^{-2}\,du$

$= \frac{1}{2}\ln|u| + \frac{1}{2}u^{-1} + C$

$= \frac{1}{2}\ln(1+x^2) + \frac{1}{2(1+x^2)} + C$

B3. *[15 marks]*

Evaluate $\displaystyle\int_2^3 \frac{x+1}{x(x-1)^2}\,dx$.

As a preliminary, we decompose $\dfrac{x+1}{x(x-1)^2}$ into partial fractions by finding A, B, and C so that

$$\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{x+1}{x(x-1)^2} \quad \text{for all } x \neq 0,\ x \neq 1.$$

That is, $A(x-1)^2 + Bx(x-1) + Cx = x+1$ for all real x. In particular, when $x=0$, $A=1$; when $x=1$, $C=2$; and when $x=2$ (for instance), $A + 2B + 2C = 3$, $2B = 3 - A - 2C = -2$, and $B = -1$.

$$\text{So } \int_2^3 \frac{x+1}{x(x-1)^2}\,dx = \int_2^3 \frac{1}{x} - \frac{1}{x-1} + \frac{2}{(x-1)^2}\,dx$$

$$= \left[\ln\frac{x}{x-1} - \frac{2}{x-1}\right]_2^3 = \left[\ln\tfrac{3}{2} - 1\right] - \left[\ln 2 - 2\right]$$

$$= 1 + \ln\tfrac{3}{2} - \ln 2 = 1 + \ln\tfrac{3}{4}$$

B4. *[15 marks]*

Over the next 7 years the profits of a business at time t (in years) are estimated to be $40,000\,e^{0.03t}$ dollars per year. The business is to be sold at a price equal to the present value of these future profits. At what price should the business be sold if interest is compounded continuously at 8%?

In dollars, the price should be

$$\int_0^7 40{,}000\, e^{.03t} \cdot e^{-.08t}\, dt$$

$$= \int_0^7 40{,}000\, e^{-.05t}\, dt$$

$$= -800{,}000 \left[e^{-.05t} \right]_0^7$$

$$= 800{,}000 \left(1 - e^{-.35}\right).$$