MAT 133Y1Y TERM TEST 3

Thursday, 28 July, 2011, 10:30 am – 12:30 pm

Code 1

FAMILY NAME Solutions

GIVEN NAME(S)

STUDENT NO.

SIGNATURE

GRADER'S REPORT	
Question	Mark
MC/40	
B1/15	
B2/15	
B3/15	
B4/15	
TOTAL	

NOTE:

1. **Aids Allowed:** Calculator with empty memory, to be supplied by student. Absolutely no graphing calculators allowed.
2. **Instructions:** Fill in the information on this page and ensure that the test contains 10 pages.
3. This test has 10 multiple choice questions worth 4 marks each and 4 written-answer questions worth 15 marks each.

   For the **multiple choice questions** indicate your answers by circling the appropriate letters (**A**, **B**, **C**, **D**, or **E**) on **this page (page 1)**. A multiple choice question left blank on **this page**, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the **written-answer questions**, present your solutions in the spaces provided. Use the multiple choice question pages or the back of any of the pages for rough work, for any of the questions.

ANSWER BOX FOR PART A

Circle the correct answer.

1.	**A**	**B**	**C**	**D**	**E**
2.	**A**	**B**	**C**	**D**	**E**
3.	**A**	**B**	**C**	**D**	**E**
4.	**A**	**B**	**C**	**D**	**E**
5.	**A**	**B**	**C**	**D**	**E**
6.	**A**	**B**	**C**	**D**	**E**
7.	**A**	**B**	**C**	**D**	**E**
8.	**A**	**B**	**C**	**D**	**E**
9.	**A**	**B**	**C**	**D**	**E**
10.	**A**	**B**	**C**	**D**	**E**

PART A. Multiple Choice

1. *[4 marks]*

Which of the following expressions is the differential approximation to $(64+h)^{\frac{2}{3}}$, for $|h|$ small?

A $16+h^{\frac{2}{3}}$

B $16+\frac{8}{3}h$

C $16+\frac{2}{3}h^{-\frac{1}{3}}$

D $16+\frac{2}{3}h$

E $16+\frac{1}{6}h$

Let $f(x)=x^{\frac{2}{3}}$, so $f'(x)=\frac{2}{3}\cdot x^{-\frac{1}{3}}$,

$f(64)=16$, and $f'(64)=\frac{1}{6}$

$$(64+h)^{\frac{2}{3}}=f(64+h)\approx f(64)+f'(64)\cdot h$$

2. *[4 marks]*

The marginal cost function for a certain product is $\dfrac{dc}{dq}=20-\dfrac{100}{(q+5)^2}$. When 45 units are produced the total cost is \$945. Then the total cost when 95 units are produced is

A \$1944

B \$945

C \$1273

D \$2154

E \$1901

Method ①

$$c(q)=\int 20-\frac{100}{(q+5)^2}\,dq=20q+\frac{100}{q+5}+K$$

Since $c(45)=945$,

$$945=20\cdot 45+\frac{100}{45+5}+K=902+K, \text{ so } K=43.$$

Then $c(q)=20q+\dfrac{100}{q+5}+43$ and $c(95)=20\cdot 95+\dfrac{100}{95+5}+43$

Method ② $c(95)=945+\displaystyle\int_{45}^{95} 20-\frac{100}{(q+5)^2}\,dq$

$$=945+\left[20q+\frac{100}{q+5}\right]_{45}^{95}$$

3. *[4 marks]*

If $f''(x) = 12x^2 - 2$, $f(1) = 2$, and $f'(1) = -1$, then $f(0) =$

A -1

B 5

C -2

D 0

E -3

$f'(x) = \int 12x^2 - 2\,dx = 4x^3 - 2x + C$

where C is determined by $f'(1) = -1$:

$-1 = 4 - 2 + C$; $C = -3$

Then $f(x) = \int f'(x)\,dx = \int 4x^3 - 2x - 3\,dx$

$= x^4 - x^2 - 3x + K$ where $K = f(0)$ is determined

by $f(1) = 2$: $2 = 1^4 - 1^2 - 3 \cdot 1 + K$

4. *[4 marks]*

If $f(x) = \displaystyle\int_1^x e^{(t^2)}\,dt$, then $f'(4) =$

A $e^{16} - e$

B e^{8}

C e^{16}

D $8e^{16}$

E $e^{\frac{64}{3}}$

By the fundamental theorem of calculus,

$f'(4) = e^{(t^2)}$, evaluated at $t = 4$.

5. *[4 marks]*

$$\int_0^3 x\sqrt{x^2+16}\,dx =$$

A $\frac{61}{3}$

B $\frac{125}{3}$

C $\frac{61}{2}$

D $\frac{125}{2}$

E 61

Let $u = x^2+16$ and when $x=0, u=16$

$du = 2x\,dx$ when $x=3, u=25$

$\frac{1}{2}du = x\,dx$

The integral is $\int_{16}^{25} u^{\frac{1}{2}}\left(\frac{1}{2}du\right)$

$= \left[\frac{1}{3}u^{\frac{3}{2}}\right]_{16}^{25} = \frac{125}{3} - \frac{64}{3}$

6. *[4 marks]*

Let $f(x) = \begin{cases} 1 & \text{if } -3 \le x < -1 \\ x & \text{if } -1 \le x \le 3 \end{cases}$. Then $\int_{-3}^{3} f(x)\,dx =$

A 4

B 3

C 7

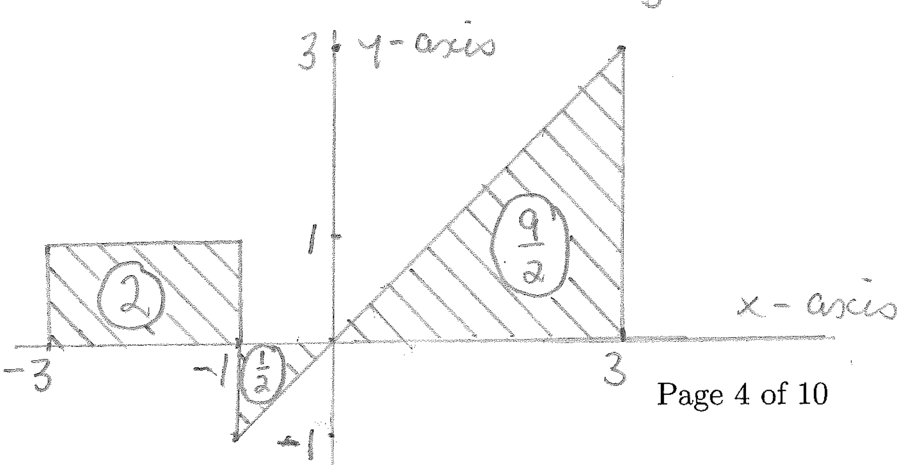
D 6

E 5

$\int_{-3}^{-1} f(x)\,dx + \int_{-1}^{3} f(x)\,dx$

$= \int_{-3}^{-1} 1\,dx + \int_{-1}^{3} x\,dx$

$= [x]_{-3}^{-1} + \left[\frac{x^2}{2}\right]_{-1}^{3}$



$= [-1-(-3)] + \left[\frac{9}{2} - \frac{1}{2}\right]$

$= 2 + 4$

7. *[4 marks]*

Let a be a fixed positive number. Which integral equals the area of the region bounded by the graph of $y = x^2$ and the lines $y = 0$ and $x = a$?

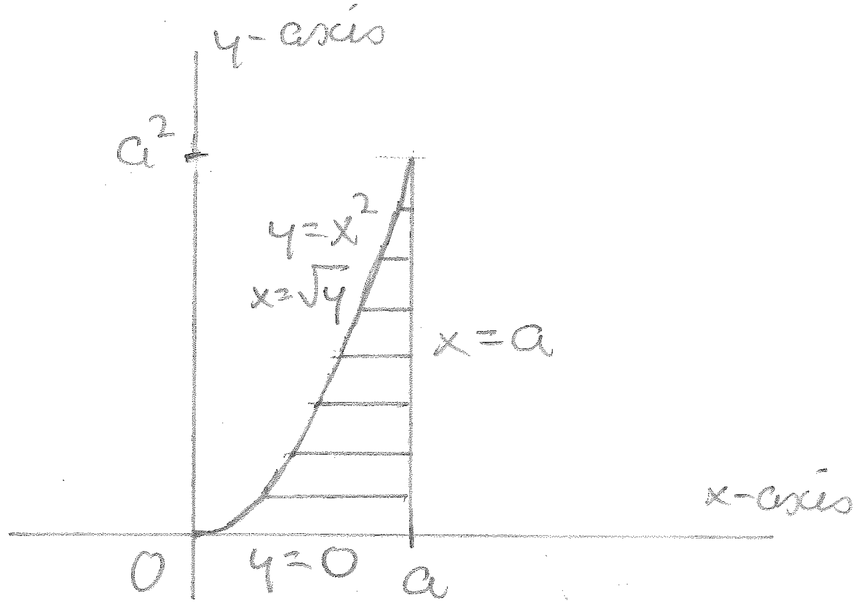
A $\int_0^{a^2} \sqrt{y}\, dy$

B $\int_0^{a^2} a - \sqrt{y}\, dy$

C $\int_0^{a} a - \sqrt{y}\, dy$

D $\int_0^{a^2} y^2\, dy$

E $\int_0^{a} \sqrt{y}\, dy$



8. *[4 marks]*

The profit from a business at time t is estimated to accrue at a continuous rate of \$20,000$e^{0.005t}$ per year. If annual interest is 6% compounded continuously, then the present value of all profit for the next 5 years (to the nearest \$10) is

A \$87,430

B \$87,490

C \$87,470

D \$87,450

E \$87,510

$$\int_0^5 20{,}000\, e^{.005t} e^{-.06t}\, dt$$

$$= 20{,}000 \int_0^5 e^{-.055t}\, dt$$

$$= 20{,}000 \left[\frac{e^{-.055t}}{-.055} \right]_0^5$$

$$= \frac{20{,}000}{.055} \left[e^{-.055t} \right]_5^0 = \frac{4{,}000{,}000}{11} \left[1 - e^{-.275} \right]$$

$$= 87{,}428.32$$

9. *[4 marks]*

If r and a are positive constants, then the average value of $y = x^r$ on the interval $[0, a]$ is

A $\dfrac{a^{r+1}}{r(r+1)}$

B $\dfrac{a^{r+1}}{r}$

C $\dfrac{a^r}{r(r+1)}$

D $\dfrac{a^r}{r+1}$

E $\dfrac{a^{r+1}}{r+1}$

$$\frac{1}{a}\int_0^a x^r\,dr$$

$$= \frac{1}{a}\left[\frac{x^{r+1}}{r+1}\right]_0^a$$

$$= \frac{1}{a}\,\frac{a^{r+1}}{r+1}$$

10. *[4 marks]*

$\displaystyle\int_e^{\infty} \frac{\ln x}{x}\,dx$

A $= 1$

B $= \frac{1}{2}$

C diverges

D $= \frac{1}{2}e$

E $= -\frac{1}{2}e$

$$= \lim_{b\to\infty}\int_e^b \ln x\,\frac{dx}{x} = \lim_{b\to\infty}\int_1^{\ln b} u\,du$$

(where $u = \ln x$, $du = \frac{dx}{x}$,
$u = 1$ when $x = e$, $u = \ln b$ when $x = b$)

$$\text{So } \int_e^{\infty}\frac{\ln x}{x}\,dx = \lim_{b\to\infty}\left[\frac{u^2}{2}\right]_1^{\ln b}$$

$$= \lim_{b\to\infty}\left[\frac{(\ln b)^2}{2} - \frac{1}{2}\right]$$

This limit diverges because as $b\to\infty$, $\ln b \to \infty$.

PART B. Written-Answer Questions
SHOW YOUR WORK.

B1. *[15 marks]*

Find the total area of the region(s) bounded by the graphs of

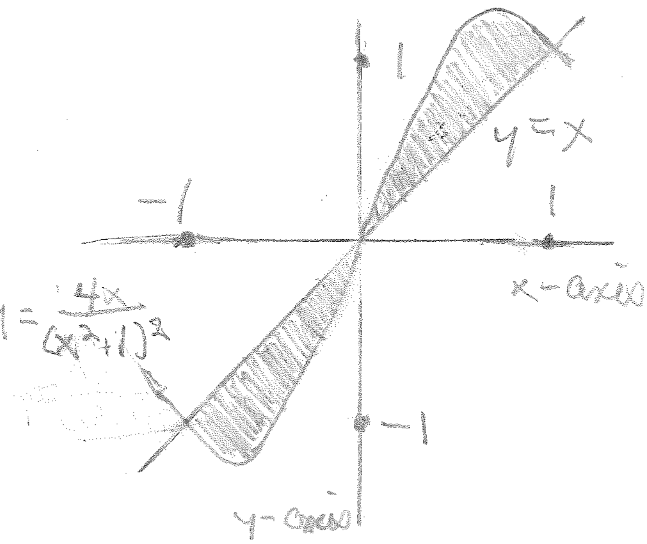
$$y = \frac{4x}{(x^2+1)^2} \text{ and } y = x\,.$$

(x, y) is an intersection point iff $y = x$ and:

$\frac{4x}{(x^2+1)^2} = x$ iff $\left(\frac{4}{(x^2+1)^2} - 1\right)x = 0$

iff $(x^2+1)^2 = 4$ or $x = 0$

iff $x = -1$ or $x = 0$ or $x = 1$



Total area is $\int_{-1}^{0} x - \frac{4x}{(x^2+1)^2}\,dx + \int_0^1 \frac{4x}{(x^2+1)^2} - x\,dx$

(or $2\int_0^1 \frac{4x}{(x^2+1)^2} - x\,dx$, in view of the fact that the two regions have the same area).

We choose the alternative which requires only one integral to be evaluated. Let $u = x^2+1$, so $du = 2x\,dx$, $4x\,dx = 2\,du$, and when $x=0$, $u=1$; when $x=1$, $u=2$.

Total area $= 2\int_1^2 \frac{2\,du}{u^2} - 2\int_0^1 x\,dx = 2\left[-\frac{2}{u}\right]_1^2 - 2\left[\frac{x^2}{2}\right]_0^1$

$= 2\left[-\frac{2}{2} + \frac{2}{1}\right] - 2\left[\frac{1^2}{2} - \frac{0^2}{2}\right] = 2\cdot 1 - 2\cdot\frac{1}{2} = 1$

B2.(a) *[8 marks]*

It is known that in a certain city the scientific workstation computer industry has the daily demand equation $f(q) = -2000q + 26000$ and the daily supply equation $g(q) = 2000q^2 + 2000$, where $p = f(q)$ and $p = g(q)$ are in dollars. Under market equilibrium, find the producers' surplus.

Let q_0 denote the equilibrium production / demand:

$2000\, q_0^2 + 2000 = -2000\, q_0 + 26000$, so $q_0^2 + q_0 - 12 = 0$

and $q_0 = 3$ (NOT $q_0 = -4$ because $-4 < 0$).

The equilibrium price (p_0) is then $f(q_0) = g(q_0)$

$= 20000$.

Producers' surplus $= \int_0^3 20000 - (2000\, q^2 + 2000)\, dq$

$= 2000 \int_0^3 9 - q^2 = 2000 \left[9q - \frac{q^3}{3} \right]_0^3$

$= 2000\,(27 - 9) = 36000$ (dollars)

B2.(b) *[7 marks]*

Find $\int x \ln x \, dx$.

Integration by parts:

Let $u = \ln x$, $dv = x\,dx$, so $du = \frac{dx}{x}$, $v = \frac{x^2}{2}$,

and $\int x \ln x\, dx = \frac{1}{2} x^2 \ln x - \int \frac{x^2}{2} \frac{dx}{x}$

$= \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C$

B3. *[15 marks]*

Evaluate $\displaystyle\int_2^\infty \frac{4x}{(x+1)(x-1)^2}\,dx$.

Partial fractions:

we find A, B, and C such that

$$\frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{4x}{(x+1)(x-1)^2} \text{ for all } x \neq -1,\ x \neq 1.$$

That is, $A(x-1)^2 + B(x+1)(x-1) + C(x+1) = 4x$ for all x.

At $x=-1$, $A \cdot 4 = -4$ so $A = -1$.

At $x = 1$, $C \cdot 2 = 4$ so $C = 2$.

At $x = 0$, $A - B + C = 0$, so $B = 1$.

$$\int_2^\infty \frac{4x}{(x+1)(x-1)^2}\,dx = \lim_{b\to\infty} \int_2^b -\frac{1}{x+1} + \frac{1}{x-1} + \frac{2}{(x-1)^2}\,dx$$

$$= \lim_{b\to\infty} \left[-\ln|x+1| + \ln|x-1| - \frac{2}{x-1}\right]_2^b$$

$$= \lim_{b\to\infty} \left[\left(-\ln(b+1) + \ln(b-1) - \frac{2}{b-1}\right) + \left(\ln 3 - \ln 1 + \frac{2}{1}\right)\right]$$

$$= \lim_{b\to\infty} \left[\left(\ln \frac{b-1}{b+1}\right) - \frac{2}{b-1}\right] + 2 + \ln 3$$

$$= \lim_{b\to\infty} \left[\left(\ln \frac{1-b^{-1}}{1+b^{-1}}\right) - \frac{2}{b-1}\right] + 2 + \ln 3 = 2 + \ln 3$$

B4. *[15 marks]*

Find $y(x)$, defined for all $x > 1$ and expressed **explicitly** in terms of x, such that $(x-1)yy' = 1$ and $y(2) = 1$.

Separate variables: $y\,dy = \dfrac{dx}{x-1}$

General solution (by integrating both sides):

$$\frac{y^2}{2} = C + \ln(x-1)$$

$\left(\text{since } x>1,\ x-1>0\right)$

$y(2)=1$, so C satisfies $\dfrac{1^2}{2} = C + \ln(2-1)$,

$C = \dfrac{1}{2}$, and this initial value problem

has particular solution $\dfrac{y^2}{2} = \dfrac{1}{2} + \ln(x-1)$

Explicitly, $y(x) = \boxed{\sqrt{1+2\ln(x-1)}}$

$= \boxed{\sqrt{1+\ln(x-1)^2}}$

(y can never be 0 since $(x-1)yy'$ would then be 0, not 1 so y is always >0)