MAT 133Y1Y TERM TEST 3

Thursday, 26 July, 2012, 10:30 am – 12:30 pm

Code 1

FAMILY NAME Solutions

GIVEN NAME(S) ______________________

STUDENT NO. ______________________

SIGNATURE ______________________

GRADER'S REPORT	
Question	Mark
MC/40	
B1/15	
B2/15	
B3/15	
B4/15	
TOTAL	

NOTE:

1. **Aids Allowed:** Calculator with empty memory, to be supplied by student. Absolutely no graphing calculators allowed.
2. **Instructions:** Fill in the information on this page and ensure that the test contains 10 pages.
3. This test has 10 multiple choice questions worth 4 marks each and 4 written-answer questions worth 15 marks each.

   For the **multiple choice questions** indicate your answers by circling the appropriate letters (**A**, **B**, **C**, **D**, or **E**) on **this page (page 1)**. A multiple choice question left blank on **this page**, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the **written-answer questions**, present your solutions in the spaces provided. Use the multiple choice question pages or the back of any of the pages for rough work, for any of the questions.

ANSWER BOX FOR PART A

Circle the correct answer.

1.	A	B	C	D	E
2.	A	B	C	D	E
3.	A	B	C	D	E
4.	A	B	C	D	E
5.	A	B	C	D	E
6.	A	B	C	D	E
7.	A	B	C	D	E
8.	A	B	C	D	E
9.	A	B	C	D	E
10.	A	B	C	D	E

PART A. Multiple Choice

1. *[4 marks]*

   If differential approximation is used, then for $h \neq 0$ with $|h|$ small, $\ln(e+h)$ is approximately equal to

   A $1 + \ln h$

   B $1 + eh$

   C $1 + e^{-1}h$

   D $1 + h^{-1}$

   E $1 + h$

   Let $f(x) = \ln x$, $f'(x) = x^{-1}$

   Differential approximation:

   $f(e+h) \approx f(e) + f'(e)h$

   That is, $\ln(e+h) \approx \ln(e) + e^{-1} \cdot h$

2. *[4 marks]*

   If the marginal revenue for a certain good is $\dfrac{dr}{dq} = e^{-\frac{1}{2}q}$, then $r(\ln 9) =$

   A $\frac{8}{3}$

   B 2

   C $\frac{5}{3}$

   D $\frac{4}{3}$

   E $\frac{7}{3}$

   $r(q) = \int e^{-\frac{1}{2}q}\,dq = -2e^{-\frac{1}{2}q} + C$

   $r = 0$ when $q = 0$:

   $0 = -2e^{0} + C$

   You may assume that $r(0) = 0$.

   So $C = 2$, $r(q) = -2e^{-\frac{1}{2}q} + 2$

   $r(\ln 9) = -2\left(e^{\ln 9}\right)^{-\frac{1}{2}} + 2$

3. *[4 marks]*

If $f''(x) = 24x^2$, $f(1) = 3$, and $f'(1) = 5$, then $f(0) =$

A 3

B 4

C 5

D 1

E 2

$f'(x) = \int 24x^2\,dx = 8x^3 + C_1$

$f'(1) = 5: \quad 5 = 8 + C_1$ so $C_1 = -3$

$f'(x) = 8x^3 - 3$

$f(x) = \int 8x^3 - 3\,dx = 2x^4 - 3x + C_0$

$f(1) = 3: \quad 3 = 2 - 3 + C_0$ so $C_0 = 4$

$f(x) = 2x^4 - 3x + 4$ and $f(0) = 4$

4. *[4 marks]*

If $f(x) = \int_0^x \sqrt{1+t^3}\,dt$, then $f'(2) =$

A 0

B 4

C 3

D 1

E 2

Let $g(t) = \sqrt{1+t^3}$

By the fundamental theorem of calculus,

$f'(x) = g(x)$.

So $f'(2) = g(2) = \sqrt{1+2^3}$

5. *[4 marks]*

$$\int_0^1 (x^2+1)^2\, dx =$$

A $\frac{5}{3}$

B $\frac{32}{15}$

C 2

D $\frac{8}{5}$

E $\frac{28}{15}$

$$\int_0^1 x^4 + 2x^2 + 1\, dx$$

$$= \left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_0^1$$

$$= \frac{3}{15} + \frac{10}{15} + \frac{15}{15}$$

6. *[4 marks]*

$$\int_0^1 x(x^2+1)^4\, dx =$$

A $\frac{14}{5}$

B $\frac{7}{2}$

C $\frac{27}{10}$

D $\frac{31}{10}$

E 3

(By substitution:
Let $u = x^2+1$, $du = 2x\,dx$, so $x\,dx = \frac{du}{2}$
When $x=0$, $u=1$. When $x=1$, $u=2$.)

$$\int_0^1 x(x^2+1)^4 dx = \int_1^2 u^4 \frac{du}{2} = \left[\frac{u^5}{10}\right]_1^2$$

$$= \frac{2^5}{10} - \frac{1^5}{10}$$

7. *[4 marks]*

If $f(x) = \begin{cases} 2 & \text{if } -1 \le x \le 3 \\ 10 - 2x & \text{if } 3 < x \le 6 \end{cases}$ then $\int_{-1}^{6} f(x)\,dx$

A $= 11$

B $= 10$

C $= 13$

D is undefined

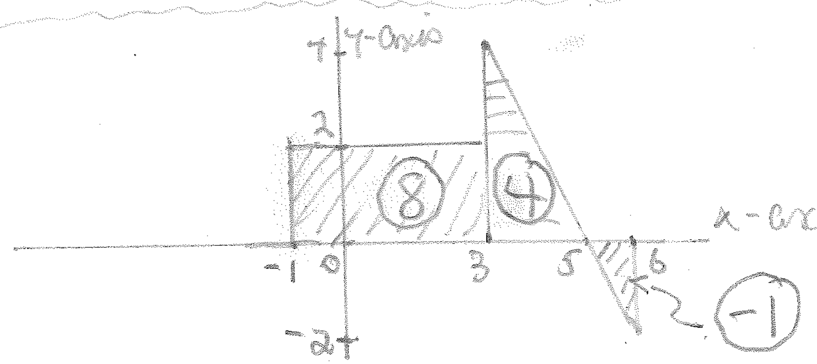
E $= 12$

$$= \int_{-1}^{3} f(x)\,dx + \int_{3}^{6} f(x)\,dx$$

$$= \int_{-1}^{3} 2\,dx + \int_{3}^{6} 10 - 2x\,dx$$

$$= \Big[2x\Big]_{-1}^{3} + \Big[10x - x^2\Big]_{3}^{6} = [6 - (-2)] + [24 - 21]$$

Alternate solution:



8. *[4 marks]*

The total area of the regions bounded by the x-axis, the line $x = -2$, and the graph of $f(x) = x^2 - 2x$ is

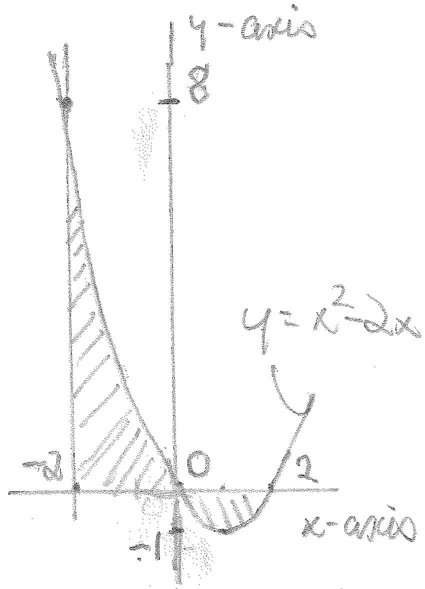
A 6

B 9

C 5

D 8

E 7

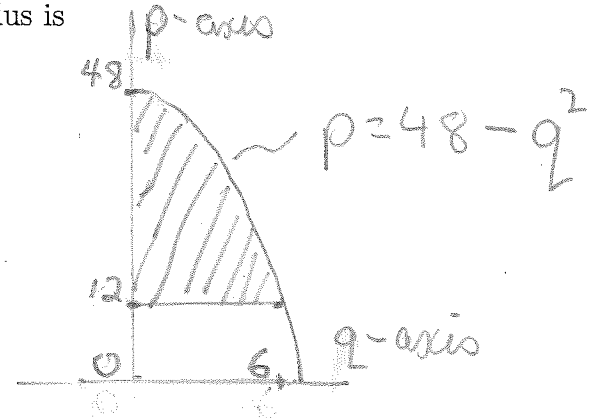


$$\int_{-2}^{0} x^2 - 2x\,dx + \int_{0}^{2} 2x - x^2\,dx = \left[\frac{x^3}{3} - x^2\right]_{-2}^{0} + \left[x^2 - \frac{x^3}{3}\right]_{0}^{2}$$

$$= -\left(-\frac{8}{3} - 4\right) + \left(4 - \frac{8}{3}\right)$$

9. *[4 marks]*

If a good has demand function $p = 48 - q^2$ and supply function $p = 2q$, then its consumers' surplus is



A 128

B 192

C 144

D 120

E 160

Hint: Note that the equilibrium point is $(q, p) = (6, 12)$.

$$CS = \int_0^6 (48-q^2) - 12\, dq = \int_0^6 36 - q^2\, dq$$

$$= \left[36q - \frac{q^3}{3}\right]_0^6 = 216 - 72$$

10. *[4 marks]*

$$\int_e^{\infty} \frac{dx}{x(\ln x)^2}$$

A $= e^{-1}$

B diverges

C $= e$

D $= e^{-2}$

E $= 1$

Substitution: Let $u = \ln x$.

Then $du = \frac{dx}{x}$ and

$$\int \frac{dx}{x(\ln x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} + C$$

$$= -\frac{1}{\ln x} + C$$

$$\int_e^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{b \to \infty} \left[-\frac{1}{\ln x}\right]_e^b$$

$$= \lim_{b \to \infty} \left[-\frac{1}{\ln b} - \left(-\frac{1}{\ln e}\right)\right] = \frac{1}{\ln e}$$

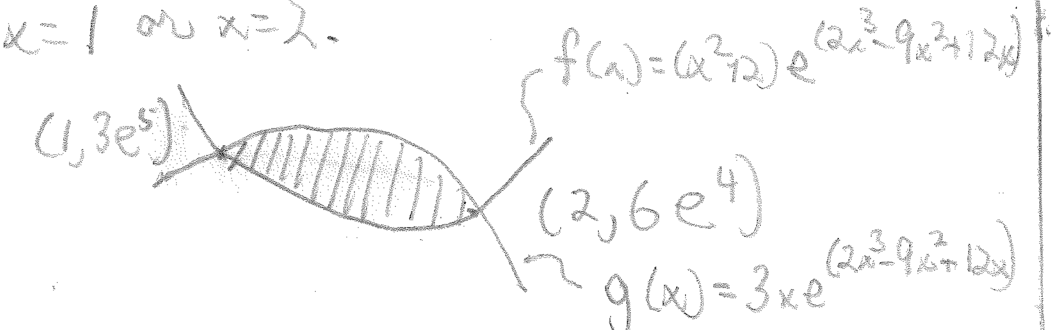
PART B. Written-Answer Questions
SHOW YOUR WORK.

B1. *[15 marks]*

Find the area of the region in the xy-plane which is bounded by the graphs of $y = (x^2+2)e^{(2x^3-9x^2+12x)}$ and $y = 3xe^{(2x^3-9x^2+12x)}$.

The graphs intersect at (x,y) provided $(x^2+2)e^{(2x^3-9x^2+12x)} = 3xe^{(2x^3-9x^2+12x)}$ and both $=y$.
That is, by cancelling $e^{(2x^3-9x^2+12x)} \neq 0$, $x^2+2=3x$, $x^2-3x+2=0$, $(x-1)(x-2)=0$, and intersections occur when $x=1$ or $x=2$.

[a rough diagram



]

To establish that $g(x) \geq f(x)$ on $[1,2]$, note that $-x^2+3x-2 = (x-1)(2-x) \geq 0$ on $[1,2]$ so that $3x \geq x^2+2$ (while the exponential is positive).
The shaded region has area $\int_1^2 (3x-(x^2+2))e^{(2x^3-9x^2+12x)}\,dx$.

Let $u = 2x^3-9x^2+12x$. Then $du = (6x^2-18x+12)\,dx$ and $(3x-(x^2+2))\,dx = (-x^2+3x-2)\,dx = -\frac{1}{6}\,du$.
When $x=1$, $u=5$. When $x=2$, $u=4$.
So the shaded area is $\int_5^4 -\frac{1}{6}e^u\,du$

$= \left[-\frac{1}{6}e^u\right]_5^4 = \frac{e^5-e^4}{6}$.

B2. *[15 marks]*

At time t, where $0 \le t \le 20$, cash flows into an account at the rate $1000t$ dollars per year. What is the present value of the cash flow if it ends in 20 years and the account always earns 4% compounded continuously?

The present value is $\int_0^{20} 1000te^{-.04t}\,dt$.

(Integration by parts: Let $u = 1000t$, $dv = e^{-.04t}dt$, so $du = 1000\,dt$, $v = -25e^{-.04t}$)

Then $\int_0^{20} 1000te^{-.04t}\,dt$

$= \left[-25{,}000te^{-.04t}\right]_0^{20} - \int_0^{20} -25000e^{-.04t}\,dt$

$= (-25{,}000)\left[te^{-.04t} + 25e^{-.04t}\right]_0^{20}$

$= (-25{,}000)(45e^{-.8} - 25)$

$= (25{,}000)(25 - 45e^{-.8})$

(dollars)

$= \$119{,}504.92$

B3. *[15 marks]*

Find the average value, on $[2,4]$, of $f(x) = \dfrac{x^2-2}{x^2(x-1)}$.

For a partial fraction decomposition of $f(x)$, we seek A, B, and C such that

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{x^2-2}{x^2(x-1)} \text{ for } x \neq 0, 1.$$

That is,

$$Ax(x-1) + B(x-1) + Cx^2 = x^2-2 \text{ for all } x.$$

When $x=0$, we find $B(-1) = -2$ and $\underline{B=2}$.

when $x=1$, $\underline{C = 1-2 = -1}$.

when $x=-1$ (say), $A(-1)(-2) + B(-2) + C = -1$,

$2A - 2B + C = -1$, $2A - 2\cdot 2 + (-1) = -1$, and $\underline{A=2}$.

The average value is

$$\frac{1}{4-2}\int_2^4 f(x)\,dx = \frac{1}{2}\int_2^4 \frac{2}{x} + \frac{2}{x^2} - \frac{1}{x-1}\,dx$$

$$= \frac{1}{2}\left[2\ln x - \frac{2}{x} - \ln(x-1)\right]_2^4$$

$$= \frac{1}{2}\left[\left(2\ln 4 - \frac{1}{2} - \ln 3\right) - \left(2\ln 2 - 1 - \cancel{\ln 1}\right)\right]$$

$$= \frac{1}{4} + \ln 2 - \frac{1}{2}\ln 3$$

$(2\ln 4 = 2\ln 2^2 = 2\cdot 2\ln 2)$

B4. *[15 marks]*

Find $y(x)$ **explicitly** in terms of x such that $\frac{dy}{dx} = x^2 e^{3y}$ and $y(0) = -1$. You may assume that $x < e$.

Separate variables: $e^{-3y}dy = x^2 dx$

Integrate $\int e^{-3y}dy = \int x^2 dx$:

$$-\frac{1}{3}e^{-3y} = \frac{1}{3}x^3 + C$$

To determine C, $y = -1$ when $x = 0$:

$$-\frac{1}{3}e^{-3(-1)} = 0 + C, \text{ so } C = -\frac{1}{3}e^3,$$

and $-\frac{1}{3}e^{-3y} = \frac{1}{3}x^3 - \frac{1}{3}e^3$

To solve for $y(x)$ explicitly,

$$e^{-3y} = e^3 - x^3$$

$$-3y = \ln(e^3 - x^3)$$

$$y(x) = -\frac{1}{3}\ln(e^3 - x^3)$$