MAT 133Y1Y TERM TEST 1

Thursday, 6 June, 2013, 7:10 pm – 9:10 pm

Code 1

FAMILY NAME ______ Solutions

GIVEN NAME(S) ______

STUDENT NO. ______

SIGNATURE ______

GRADER'S REPORT	
Question	Mark
MC/40	
B1/15	
B2/15	
B3/15	
B4/15	
TOTAL	

NOTE:

1. **Aids Allowed:** Calculator with empty memory, to be supplied by student. Absolutely no graphing calculators allowed.
2. **Instructions:** Fill in the information on this page and ensure that the test contains 10 pages.
3. This test has 10 multiple choice questions worth 4 marks each and 4 written-answer questions worth 15 marks each.

   For the **multiple choice questions** indicate your answers by circling the appropriate letters (**A**, **B**, **C**, **D**, or **E**) on **this page (page 1)**. A multiple choice question left blank on **this page**, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the **written-answer questions**, present your solutions in the spaces provided. Use the multiple choice question pages or the back of any of the pages for rough work, for any of the questions.

ANSWER BOX FOR PART A

Circle the correct answer.

1.	**A**	**B**	C	**D**	**E**
2.	**A**	**B**	C	**D**	**E**
3.	**A**	**B**	C	**D**	**E**
4.	**A**	**B**	C	**D**	**E**
5.	**A**	**B**	C	**D**	**E**
6.	**A**	**B**	C	**D**	**E**
7.	**A**	**B**	C	**D**	**E**
8.	**A**	**B**	C	**D**	**E**
9.	**A**	**B**	C	**D**	**E**
10.	**A**	**B**	C	**D**	**E**

PART A. Multiple Choice

1. *[4 marks]*

How much does a perpetuity cost (to the nearest $10) if its payments are **annual** and $10,000 each, when interest is 8% compounded **semiannually**?

A $122,550
B $119,670
C $125,000
D $115,430
E $121,340

effective annual interest

$= (1+\frac{8\%}{2})^2 - 1 = .0816$

$$\frac{\$10,000}{.0816} = \$122549.02$$

2. *[4 marks]*

A $10,000 debt is to be repaid with a payment of $X in ~~2 years~~ now, a payment of $2X in 15 months, and a final payment of $X in 2 years. If interest is 6% compounded continuously, then (to the nearest dollar) $X =

A $2671
B $2667
C $2674
D $2672
E $2669

$$\$X + \$2X e^{-.06 \cdot \frac{15}{12}} + \$X e^{-.06 \cdot 2} = \$10,000$$

$$\$X = \frac{\$10,000}{1 + 2e^{-.075} + e^{-.12}}$$

$$= \$2672.08$$

3. *[4 marks]*

A car is advertised for $100 down and $500 per month for 36 months. Suppose that the interest rate is 12% per annum, compounded monthly. What is the **increase** in the down payment that you should offer (to the nearest dollar), in order to reduce the monthly payments to $300 per month for 36 months?

- **A** $6032
- **B** $6019
- **C** $7200
- **D** $6027
- **E** $6022

monthly interest $= \frac{12\%}{12} = .01$

Each payment is to be reduced by $200

$$\$200\, a_{\overline{36}|.01} = \$200 \, \frac{1-(1.01)^{-36}}{.01}$$

$$= \$6021.5009$$

4. *[4 marks]*

If monthly deposits of $100 each are made into an account earning 4.8% compounded monthly, on which deposit will the account first exceed $5000?

- **A** the 49^{th} deposit
- **B** the 46^{th} deposit
- **C** the 44^{th} deposit
- **D** the 48^{th} deposit
- **E** the 47^{th} deposit

monthly interest $= \frac{4.8\%}{12} = .004$

Let n = # of deposits required to accumulate $5000

$$\$100\, s_{\overline{n}|.004} = \$5000$$

$$\frac{(1.004)^n - 1}{.004} = \frac{\$5000}{\$100} = 50. \text{ So } (1.004)^n - 1 = .004 \cdot 50 = .2$$

$$(1.004)^n = 1.2, \; n \ln(1.004) = \ln(1.2), \text{ and } n = \frac{\ln(1.2)}{\ln(1.004)}$$

$$= 45.67\ldots$$

5. *[4 marks]*

What is the market price of a $100 bond which has **semiannual** interest payments (coupons) worth $1.70 each, **annual** yield 4%, and 6 years to maturity?

A $96.83

B $99.10

C $95.97

D $98.35

E $97.18

semiannual yield $= \frac{4\%}{2} = .02$

$$\$100\,(1.02)^{-12} + \$1.70\,\frac{1-(1.02)^{-12}}{.02}$$

$$= \$100\,(1.02)^{-12} + \$85\,(1-(1.02)^{-12})$$

$$= \$15\,(1.02)^{-12} + \$85$$

$$= \$96.83$$

6. *[4 marks]*

If $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, then $A^6 =$

A $\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$

B $\begin{bmatrix} -8 & 0 \\ 0 & 8 \end{bmatrix}$

C $\begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}$

D $\begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix}$

E $\begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix}$$

$$A^6 = \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix}$$

7. *[4 marks]*

Find **all** pairs (x, y) so that the matrix $\begin{bmatrix} 1 & 0 & 1 \\ x & y & 0 \end{bmatrix}$ is in row-echelon form.

A $(x, y) = (0, 0)$ and $(x, y) = (1, 0)$

B $(x, y) = (1, 1)$ only

C $(x, y) = (1, 0)$ and $(x, y) = (0, 1)$

D $(x, y) = (0, 0)$ and $(x, y) = (0, 1)$

E $(x, y) = (0, 1)$ only

8. *[4 marks]*

If $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the solution of the system

$$\begin{array}{rcrcrcl} x & & & + & 3z & = & 2 \\ x & + & y & + & z & = & 2 \\ & & 2y & - & 3z & = & 1 \end{array}$$

then $x =$

A -2

B 0

C -1

D 2

E 1

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 1 & 1 & 1 & 2 \\ 0 & 2 & -3 & 1 \end{array}\right] \approx \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & -3 & 1 \end{array}\right]$$

$$\approx \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \approx \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

9. *[4 marks]*

The system of equations

$$\begin{array}{ccccccccccc} x & + & y & + & z & + & u & + & v & = & 1 \\ & & y & + & z & + & 2u & & & = & 2 \\ & & & & z & + & u & + & 2v & \cancel{+} = & 3 \end{array}$$

has

A no solution

B a unique solution

C infinitely many solutions with one parameter

D infinitely many solutions with two parameters

E infinitely many solutions with three parameters

10. *[4 marks]*

If $A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $AX = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$ then $X =$

A $\begin{bmatrix} 2 & 3 \\ 6 & 8 \end{bmatrix}$

B $\begin{bmatrix} -1 & 5 \\ -3 & 11 \end{bmatrix}$

C $\begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

D $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

E $\begin{bmatrix} -1 & 7 \\ -2 & 10 \end{bmatrix}$

$$X = (A^{-1}A)X = A^{-1}(AX)$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ -3 & 11 \end{bmatrix}$$

PART B. Written-Answer Questions
SHOW YOUR WORK.

B1. *[15 marks]*

Tom owes Jerry two debts:
— \$1000 due now; and
— \$3000 plus interest at 5% compounded quarterly, due in 3 years.
They have agreed that the debts will be repaid in two payments:
— the first payment to be made in 4 years; and
— the second payment to be one-third the amount of the first, to be made in 5 years.
What should be the amounts of the first and second payments, if money is worth 6% compounded semiannually?

$\$3000 \longrightarrow \$3000(1.0125)^{12} \dashrightarrow \$3000(1.0125)^{12}(1.03)^{4}$

$\$1000 \dashrightarrow \$1000(1.03)^{10}$

debts: now, 3 yr, 4 yr, 5 yr (*)

payments: $\$3x \dashrightarrow \$3x(1.03)^2$, $\$x$

(*) at 5 yr = reference point

$$\$3x(1.03)^2 + \$x = \$3000(1.0125)^{12}(1.03)^4 + \$1000(1.03)^{10}$$

$$\$x = \frac{\$3000(1.0125)^{12}(1.03)^4 + \$1000(1.03)^{10}}{3(1.03)^2 + 1}$$

$= \boxed{\$1258.33}43$ (second payment)

first payment $= 3 \times \$1258.3343 = \boxed{\$3775.00}$

B2. *[15 marks]*

A $\$300,000$ mortgage is to be repaid by making equal monthly payments for 15 years, the first payment 1 month after the loan is granted. If interest is 8% per year compounded semiannually find, to within $\$0.01$:

B2.(a) *[5 marks]*

the amount of each payment

$(1.04)^{\frac{1}{6}} - 1 = .006558197$ (effective monthly rate)

$$\text{Monthly payment} = \$300{,}000 \, \frac{(1.04)^{\frac{1}{6}} - 1}{1 - (1.04)^{-30}}$$

$$= \$2844.46$$

B2.(b) *[10 marks]*

the amount of principal repaid in the 100^{th} payment

Principal outstanding just after the 99^{th} payment

is $\$300{,}000 \, \frac{(1.04)^{\frac{1}{6}} - 1}{1 - (1.04)^{-30}} \, a_{\overline{81}|(1.04)^{\frac{1}{6}} - 1}$

$$= \$300{,}000 \, \frac{1 - (1.04)^{-13.5}}{1 - (1.04)^{-30}} = \$178{,}299.73$$

Interest in the 100^{th} payment

is $\$178{,}299.73 \cdot .0065582 = \$1169.33$

Principal repaid in the 100^{th} payment

is $\$2844.46 - \$1169.33 = \$1675.13$

B3. *[15 marks]*

A 1 litre bottle is filled with a mixture of red, green, and blue liquids. The red liquid weighs 1 kg per litre, the green weighs 0.8 kg per litre, and the blue weighs 1.5 kg per litre. Moreover, the bottle contains exactly twice as much blue liquid as red, by volume. Set up and solve a system of linear equations to find the volume of each liquid used to make the mixture, if the total weight of liquid in the bottle is 1.3 kg.

Let r, g, and b represent, respectively, the volumes of red, green, and blue liquids

$$\begin{aligned} r + g + b &= 1 \\ r + .8g + 1.5b &= 1.3 \\ 2r \qquad - b &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & .8 & 1.5 & 1.3 \\ 2 & 0 & -1 & 0 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -.2 & .5 & .3 \\ 0 & -2 & -3 & -2 \end{array}\right]$$

$$\simeq \left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{2} & \frac{5}{2} \\ 0 & 1 & -\frac{5}{2} & -\frac{3}{2} \\ 0 & 0 & -8 & -5 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{16} \\ 0 & 1 & 0 & \frac{1}{16} \\ 0 & 0 & 1 & \frac{5}{8} \end{array}\right]$$

$\frac{5}{16}$ litre of red, $\frac{1}{16}$ litre of green, and $\frac{5}{8}$ litre of blue liquids.

B4(a). *[12 marks]*

Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right] \simeq \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 2 & 8 & -1 & 0 & 1 \end{array}\right]$$

$$\simeq \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right] \simeq \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -\frac{5}{2} & 4 & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right]$$

inverse

B4(b). *[3 marks]*

Use your solution to question B4.(a) to solve the system

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + 4z &= 4 \\ x + 3y + 9z &= 8 \end{aligned}$$

Note: **No** marks will be assigned to another method of solution.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$