MAT 133Y1Y TERM TEST 1

Thursday, 4 June, 2015, 6:10 pm – 8:00 pm

Code 1

FAMILY NAME ______________ Solutions

GIVEN NAME(S) ______________

STUDENT NO. ______________

SIGNATURE ______________

GRADER'S REPORT	
Question	Mark
MC/40	
B1/15	
B2/15	
B3/15	
B4/15	
TOTAL	

NOTE:

1. **Aids Allowed:** Calculator with empty memory, to be supplied by student. Absolutely no graphing calculators allowed.
2. **Instructions:** Fill in the information on this page and ensure that the test contains 10 pages.
3. This test has 10 multiple choice questions worth 4 marks each and 4 written-answer questions worth 15 marks each.

   For the **multiple choice questions** indicate your answers by circling the appropriate letters (**A**, **B**, **C**, **D**, or **E**) on **this page (page 1)**. A multiple choice question left blank on **this page**, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the **written-answer questions**, present your solutions in the spaces provided. Use the multiple choice question pages or the back of any of the pages for rough work, for any of the questions.

ANSWER BOX FOR PART A

Circle the correct answer.

1.	A	B	C	D	E
2.	A	B	C	D	E
3.	A	B	C	D	E
4.	A	B	C	D	E
5.	A	B	C	D	E
6.	A	B	C	D	E
7.	A	B	C	D	E
8.	A	B	C	D	E
9.	A	B	C	D	E
10.	A	B	C	D	E

PART A. Multiple Choice

1. *[4 marks]*

   An account with a 5% interest rate compounded continuously earns the same effective interest as an account with interest compounded semiannually at the nominal annual rate of

   A 5.11%

   B 5.03%

   C 5.06%

   D 5.08%

   E 5.14%

Let r = the nominal annual rate compounded semiannually

Ratio by which money multiplies in one year $= \left(1+\frac{r}{2}\right)^2 = e^{.05}$

$$1+\frac{r}{2} = e^{.025}$$

$$r = 2\left(e^{.025}-1\right)$$

2. *[4 marks]*

   How much money (to the nearest dollar) would an endowment fund require in order for it to provide \$40,000 per year indefinitely, if the interest rate is always 7% compounded annually?

   A \$538,291

   B \$695,026

   C \$482,206

   D \$405,390

   E \$571,428

$$\frac{\$40{,}000}{.07}$$

3. *[4 marks]*

Parents opened a college trust fund for their son on 1 September, 1999 when he started grade 1. Every year on this date, starting on 1 September, 1999, they deposited \$3000 into the fund. The last payment was made when their son started grade 12 on 1 September, 2010. If the account earns 5% compounded annually, how much money is there in the fund on 1 September, 2011?

A \$52,449.27

B \$50,138.95

C \$49,262.09

D \$47,751.38

E \$53,861.40

$$\$3000\, s_{\overline{12|}.05}\,(1.05)$$

$$= \$3000\, \frac{(1.05)^{12} - 1}{.05}\,(1.05)$$

4. *[4 marks]*

A \$3000 loan is amortized over 8 years with monthly payments of \$39.42 at an interest rate of 6% compounded monthly. The difference between the interest paid in the first and last payments is

A \$15.00

B \$14.57

C \$0

D \$15.33

E \$14.80

Monthly interest = .005

Interest in first payment

$= \$3000 \cdot .005 = \$15.00$

Outstanding principal at time of last payment $= \dfrac{\$39.42}{1.005}$

Interest in last payment $= \dfrac{\$39.42}{1.005} \cdot .005 = \$0.20$

5. *[4 marks]*

On the day after a coupon payment, the price of a \$100 bond with 10 semiannual coupon payments remaining, an annual coupon rate of 3%, and an annual yield of 4% is

A \$95.51

B \$101.73

C \$100.00

D \$97.18

E \$103.96

$$\$100 \cdot (1.02)^{-10} + \$1.50 \, \frac{1-(1.02)^{-10}}{.02}$$

6. *[4 marks]*

$$[1 \quad 1]\left(3\begin{bmatrix} -1 & 2 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}^T\right) =$$

A $[9 \quad 14]$

B $[-13 \quad 5]$

C $[6 \quad -19]$

D $[28 \quad -2]$

E $[-10 \quad -4]$

$$\begin{bmatrix} 2 & -3 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} -14 & 1 \\ -5 & 10 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 6 \\ 12 & 3 \end{bmatrix} - \begin{bmatrix} -14 & 1 \\ -5 & 10 \end{bmatrix} = \begin{bmatrix} 11 & 5 \\ 17 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 11 & 5 \\ 17 & -7 \end{bmatrix} = [28 \quad -2]$$

7. *[4 marks]*

Which of the following matrices is in row-echelon form?

A $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ the leading element of the third row is not to the right of the leading element of the second row

B $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ a row of 0's followed by a non-zero row

C $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ the first non-zero element of the second row is "2"

D $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

E $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ again, the third leading element is not to the right of the second leading element

8. *[4 marks]*

For which real number a does the system

$$\begin{array}{ccccccc} x & + & y & + & z & = & 5 \\ x & + & 2y & + & 2z & = & 9 \\ x & + & 4y & + & 4z & = & a \end{array}$$

have infinitely many solutions?

A 15

B 17

C 18

D 16

E 19

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 2 & 2 & 9 \\ 1 & 4 & 4 & a \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 4 \\ 0 & 3 & 3 & a-5 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & a-17 \end{array}\right]$$

9. *[4 marks]*

The system of linear equations

$$\begin{array}{rcrcrcrcr} 3w & - & 3x & - & 9y & - & 6z & = & 3 \\ 3w & - & 2x & - & 5y & - & 4z & = & 2 \\ 4w & - & x & & & - & z & = & 3 \end{array}$$

has

A a two-parameter family of solutions

B no solution

C a one-parameter family of solutions

D a three-parameter family of solutions

E a unique solution

$$\left[\begin{array}{cccc|c} 3 & -3 & -9 & -6 & 3 \\ 3 & -2 & -5 & -4 & 2 \\ 4 & -1 & 0 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & -3 & -2 & 1 \\ 0 & 1 & 4 & 2 & -1 \\ 0 & 3 & 12 & 7 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & a \\ 0 & 1 & 4 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

10. *[4 marks]*

Let O denote the 4×4 zero matrix and I the 4×4 identity matrix. If A is a 4×4 matrix such that

$$(A - I)(A^2 + A + I) = O$$

then $A^{-1} =$

A A^2

B A

C $A - I$

D $A^2 + I$

E $A + I$

Hint: Expand the left hand side of the equation.

$$(A-I)(A^2+A+I) = A(A^2+A+I) - I(A^2+A+I)$$
$$= (A^3+A^2+A) - (A^2+A+I) = A^3 - I$$

So $A^3 - I = O$ and $A^3 = I$. That is, $A \cdot A^2 = I$.

PART B. Written-Answer Questions
SHOW YOUR WORK.

B1. *[15 marks]*

A 15 year mortgage for \$400,000 has weekly payments with the first payment due one week after the loan is made. Interest is 5% compounded semiannually.

B1.(a) *[8 marks]* $15 \times 52 = 780$ payments

To the nearest dollar, what is the amount of each payment? Assume that 1 year equals 52 weeks.

Let $r = (1.025)^{\frac{1}{26}} - 1$ (effective weekly interest)

Let \$R = the amount of each payment

Then $\$R \, a_{\overline{780}|r} = \$400{,}000$ and

$$\$R = \$400{,}000 \frac{(1.025)^{\frac{1}{26}} - 1}{1 - (1.025)^{-\frac{780}{26}}} = \$726.35$$

B1.(b) *[7 marks]*

To the nearest dollar, what is the principal outstanding at the end of 10 years?

At the end of 10 years, 5 years = 260 weeks remain and outstanding principal

$$= \$R \, a_{\overline{260}|r} = \$167{,}261.19$$

B2. *[15 marks]*

A loan of $\$80,000$ is amortized at 9% per year compounded monthly, with monthly payments of $\$1000$ each and a smaller last payment. The first payment is due one month after the loan is made.

B2.(a) *[8 marks]*

How many payments does the debtor need to make in total?

Let n = the number of payments

Monthly interest = .0075

$$\$80{,}000 = \$1000\, a_{\overline{n}|.0075}$$

$$\text{So } 80 = \frac{1-(1.0075)^{-n}}{.0075}, \quad .6 = 1-(1.0075)^{-n},$$

$$(1.0075)^{-n} = 1-.6 = .4, \quad -n\ln(1.0075) = \ln(.4),$$

$$\text{and } n = -\frac{\ln .4}{\ln(1.0075)} = 122.6\ldots$$

123 payments

B2.(b) *[7 marks]*

To the nearest cent, how much is the last payment?

$$\$80{,}000(1.0075)^{123}$$

$$-\$1000\, s_{\overline{122}|.0075}(1.0075)$$

$$= \$630.54$$

B3. *[15 marks]*

A 20 litre can is filled with a mixture of 3 fuels: heavy hydrocarbon (which costs \$1 per litre and weighs 0.5 kilograms per litre), light hydrocarbon (which costs \$2 per litre and weighs 0.4 kilograms per litre), and alcohol (which costs \$5 per litre and weighs 0.7 kilograms per litre). If the mixture in the can costs \$38 total and weighs 10 kilograms total, find the number of litres of each fuel used to make the mixture.

Let x = # litres of heavy hydrocarbon

y = # litres of light hydrocarbon

z = # litres of alcohol

$$\begin{aligned} x + y + z &= 20 \\ x + 2y + 5z &= 38 \\ .5x + .4y + .7z &= 10 \end{aligned}$$

Solution of the system:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 1 & 2 & 5 & 38 \\ .5 & .4 & .7 & 10 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & 20 \\ 0 & 1 & 4 & 18 \\ 0 & -.1 & .2 & 0 \end{array}\right]$$

$$\simeq \left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & 18 \\ 0 & 0 & .6 & 1.8 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

11 litres heavy hydrocarbon, 6 litres light hydrocarbon, and 3 litres alcohol

B4. *[15 marks]*

Let a be any real number and let $A = \begin{bmatrix} 1 & a+1 & -2a+2 \\ -1 & 0 & 2a-2 \\ 1 & a+1 & -a+4 \end{bmatrix}$.

B4.(a) *[5 marks]*

For which values of a is A **not** invertible?

$$\left[\begin{array}{ccc|ccc} 1 & a+1 & -2a+2 & 1 & 0 & 0 \\ -1 & 0 & 2a-2 & 0 & 1 & 0 \\ 1 & a+1 & -a+4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & a+1 & -2a+2 & 1 & 0 & 0 \\ 0 & a+1 & 0 & 1 & 1 & 0 \\ 0 & 0 & a+2 & -1 & 0 & 1 \end{array}\right]$$

A is not invertible iff $a=-1$ or $a=-2$.

B4.(b) *[6 marks]*

Find A^{-1} when $a = 2$.

From the second augmented matrix above with $a=2$:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 1 & 0 \\ 0 & 0 & 4 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 4 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & 0 & \frac{1}{4} \end{array}\right]$$

A^{-1}

B4.(c) *[4 marks]*

Solve the system of linear equations:
$$\begin{array}{rcrcrcr} x & + & 3y & - & 2z & = & -3 \\ -x & & & + & 2z & = & 2 \\ x & + & 3y & + & 2z & = & 3 \end{array}.$$

Since A (with $a=2$) is the coefficient matrix of the system,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{3} \\ \frac{3}{2} \end{bmatrix}$$