MAT 133Y1Y TERM TEST #2

THURSDAY, JULY 8, 2010 7:00 - 9:00 PM

FAMILY NAME: ______________________________

GIVEN NAMES: ______________________________

STUDENT NUMBER: ______________________________

TUTORIAL ROOM: ______________________________

Aids Allowed: Calculator with empty memory, to be supplied by the student. Absolutely no graphing calculators allowed.
Instructions: This test has 10 multiple choice questions worth 4 marks each and 5 written answer questions worth a total of 60 marks. For each multiple choice question, you may do your rough work in the test booklet, but you must record your answer by circling one of the letters A, B, C, D or E which appear on the front page of the test. A multiple choice question left blank, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the written answer solutions, present your solutions in the spaces provided. Use the back of the question pages for your rough work.

GRADER'S REPORT	
Multiple Choice	/ 40
Question 11	/ 14
Question 12	/ 14
Question 13	/ 20
Question 14	/ 12
TOTAL	/100

ANSWERS FOR MULTIPLE CHOICE Circle the correct answer					
1.	**A**	**B**	**C**	**D**	**E**
2.	**A**	**B**	**C**	**D**	**E**
3.	**A**	**B**	**C**	**D**	**E**
4.	**A**	**B**	**C**	**D**	**E**
5.	**A**	**B**	**C**	**D**	**E**
6.	**A**	**B**	**C**	**D**	**E**
7.	**A**	**B**	**C**	**D**	**E**
8.	**A**	**B**	**C**	**D**	**E**
9.	**A**	**B**	**C**	**D**	**E**
10.	**A**	**B**	**C**	**D**	**E**

1. The function: $f(x) = \dfrac{3 - 2x^2}{1 + 2x + x^2}$ has asymptotes at:

A $y = -1$ and $x = -1$

(B) $y = -2$ and $x = -1$

C $y = 0$ and $x = -1$

D $y = -2$ and $x = 1$

E $y = 3$ and $x = -1$

$f(x) = \dfrac{3-2x^2}{(x+1)^2}$

$x \to -1 \quad \dfrac{3-2x^2 \to 1}{(x+1)^2 \to 0^+} \to +\infty$.

$\therefore x = -1$ is a vert. asym.

$x \to \pm\infty \quad \dfrac{3-2x^2 \to -\infty}{(x+1)^2 \to +\infty} = \dfrac{\frac{3}{x^2} - 2}{\frac{1}{x^2} + \frac{2}{x} + 1} \to -2.$ ($\frac{3}{x^2} \to 0$, $\frac{1}{x^2} \to 0$, $\frac{2}{x} \to 0$)

$\therefore y = -2$ is a hor. asym.

2. $\lim\limits_{x \to 2} (11 - 5x)^{\frac{1}{(3x-6)}} = e^{-5/3}$ ($11-5x \to 1$, $\frac{1}{3x-6} \to \pm\infty$)

A 1

B 5/3

C $e^{5/3}$

D $-5/3$

(E) $e^{-5/3}$

$\lim\limits_{x\to 2} \ln (11-5x)^{\frac{1}{3x-6}}$

$= \lim\limits_{x\to 2} \dfrac{\ln(11-5x) \to 0}{3x-6 \to 0}$

$= \lim\limits_{x\to 2} \dfrac{\frac{-5}{11-5x}}{3}$

$= -\dfrac{5}{3}.$

3. $\lim\limits_{x \to +\infty} \overset{\nearrow +\infty}{\sqrt{x^2 + x}} - \overset{\nearrow +\infty}{x} = $

$$\lim_{x\to+\infty} \frac{(\sqrt{x^2+x} - x)(\sqrt{x^2+x}+x)}{(\sqrt{x^2+x}+x)}$$

$$= \lim_{x\to+\infty} \frac{x^2+x-x^2}{\sqrt{x^2+x}+x} = \lim_{x\to+\infty} \frac{\overset{\nearrow +\infty}{x}}{\underbrace{\sqrt{x^2+x}+x}_{\searrow +\infty}}$$

$$= \lim_{x\to+\infty} \frac{1}{\sqrt{1+\underbrace{\tfrac{1}{x}}_{\searrow 0}}+1} = \frac{1}{2}.$$

[✓ A] 1/2

B 0

C 1

D −1/2

E does not exist

4. If $\dfrac{x^3 - x^2}{x + 2} \geq 0$ then

$$\overbrace{\frac{x^2(x-1)}{x+2}}^{f} \geq 0.$$

f is not defined at $x=-2$

$f = 0 \Rightarrow x = 0, 1$

$-2 \qquad 0 \qquad 1$

$f(-3)>0 \quad f(-1)<0 \quad f(\tfrac{1}{2})<0 \quad f(2)>0$

Solution: $x<-2$ or $x\geq 1$ or $x=0$

A $-2 \leq x \leq 0$ or $x \geq 1$

[✓ B] $x < -2$ or $x \geq 1$ or $x = 0$

C $x < -2$ or $0 \leq x \leq 1$

D $-2 < x \leq 1$

E $x < -2$ or $x \geq 1$

5. $\lim\limits_{h \to 0} \dfrac{[f(x+h)]^2 - [f(x)]^2}{h} = \dfrac{d}{dx}[f(x)]^2 = 2f(x)f'(x)$

A $[f'(x)]^2$

B $2[f'(x)]$

C $2[f(x)]f'(x)$

D $2[f(x)]$

E 0 (for all x)

(hint: use the limit definition of derivative)

6. If $f(x) = \sqrt{3x^2 + 2\sqrt{6x+4}}$ then the rate of change of f with respect to x when $x = 0$ is:

$$f'(x) = \frac{1}{2}\left[3x^2+2(6x+4)^{1/2}\right]^{-1/2}\left[6x+(6x+4)^{-1/2}(6)\right]$$

$$f'(0) = \frac{1}{2}\left[0+2\sqrt{4}\right]^{-1/2}\left[0+\frac{6}{\sqrt{4}}\right]$$

$$= \frac{3}{4}.$$

A 3/4

B 1/4

C 3

D 3/2

E $1/(2\sqrt{3})$

7. The function: $f(x) = x^3 - 3x$ on the interval $[-2,3]$ has:

$f'(x) = 3x^2 - 3$

$f' = 0 \Rightarrow x = \pm 1$

A one absolute maximum and one absolute minimum

B one absolute maximum and two absolute minimums

C one absolute maximum and no absolute minimum

D two absolute maximums and one absolute minimum

E two absolute maximums and two absolute minimums

$f(-2) = -2 \leftarrow$ abs min

$f(-1) = 2$

$f(1) = -2 \leftarrow$ abs min

$f(3) = 18 \leftarrow$ abs max

8. If $C = 0.8\ I - 0.4\sqrt{I}$ where I is the national income and C is the consumption, then the Marginal Propensity to Save, when $I = 16$ is:

A 0.75

B 0.3

C 0.7

D 0.36

E 0.25

$$\frac{dc}{dI} = 0.8 - 0.2 I^{-1/2}$$

$$\left.\frac{dc}{dI}\right|_{I=16} = 0.8 - \frac{0.2}{\sqrt{16}} = 0.75$$

$$MPS\Big|_{I=16} = 1 - MPC\Big|_{I=16}$$

$$= 1 - 0.75$$

$$= 0.25$$

9. If Newton's Method is used to approximate a solution of the equation: $x + \ln x = 0$ with initial estimate $x_1 = 1$, then $x_2 =$

A 1

B 1.5

C 0.567

D 0.5

E 1.567

$f(x) = x + \ln x \quad f'(x) = 1 + \frac{1}{x}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1 - \frac{f(1)}{f'(1)}$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}.$$

10. If $q = \sqrt{1200 - p}$ is the demand equation for a product, then the point elasticity of demand when 10 units are produced is:

A -2/11

B -1/238

C -2200

D -238

E -11/2

$$\eta = \frac{p}{q} \cdot \frac{dq}{dp}$$

$$= \frac{p}{\sqrt{1200-p}} \cdot \frac{-1}{2\sqrt{1200-p}} = \frac{-p}{2(1200-p)}$$

$$\eta \Big|_{q=10} = -\frac{1100}{200} = -5\tfrac{1}{2}$$

$\Downarrow$

$p = 1100$

11. Given the following function:

$$f(x) = \begin{cases} \dfrac{x^2 + x - 2}{x + 2} & \text{if } x < -2 \\ \dfrac{1 - 4^{-1/x}}{1 + 4^{-1/x}} & \text{if } -2 \le x < 0 \\ \dfrac{x + 2}{x - 2} & \text{if } x \ge 0 \end{cases}$$

(a) Evaluate the following limits: (Show all your steps)

i) $\lim\limits_{x \to -2^+} f(x) = \lim\limits_{x \to -2^+} \dfrac{1 - 4^{-1/x}}{1 + 4^{-1/x}} = \dfrac{1 - \sqrt{4}}{1 + \sqrt{4}} = \dfrac{-1}{3}$ 1

ii) $\lim\limits_{x \to -2^-} f(x) = \lim\limits_{x \to -2^-} \dfrac{(x+2)(x-1)}{x+2} = \lim\limits_{x \to -2^-} (x-1) = -3$ 3

(8)

iii) $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \dfrac{x+2}{x-2} = -1$ 1

iv) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \dfrac{1 - \overbrace{4^{-1/x}}^{\to \infty}}{1 + \underbrace{4^{-1/x}}_{\to \infty}} = \lim\limits_{x \to 0^-} \dfrac{\overbrace{4^{1/x}}^{\to 0} - 1}{\underbrace{4^{1/x}}_{\to 0} + 1} = -1$ 3

(b) By using the results of part (a), determine whether f(x) is continuous at each of the following values of x: (Justify your answer)

i) x = −2 $\quad \lim\limits_{x \to -2} f(x)$ does not exist $\therefore$ f is not continuous at $x = -2$. 2

(6) ii) x = 0 $\quad \lim\limits_{x \to 0} f(x) = -1 = f(0)$ $\therefore$ f is continuous at $x = 0$ 2

iii) x = 2 $\quad f(2) = \dfrac{4}{0}$ = undefined. $\therefore$ f is not continuous at $x = 2$. 2

12. (a) Given the relation: $x^{(x^3)}\, 3^{(-y)} = e^{(-3x)}$

Find $\frac{dy}{dx}$ when $x = 1$ by using logarithmic differentiation. (Leave answer exact)

$$\ln x^{(x^3)} + \ln 3^{(-y)} = \ln e^{(-3x)} \quad (1)$$

$$x^3 \ln x - y \ln 3 = -3x \quad (1)$$

(6) $$(*) \quad 3x^2 \ln x + \frac{1}{x} x^3 - \frac{dy}{dx} \ln 3 = -3 \quad (2)$$

$$x=1 \Rightarrow 3\ln 1 + 1 - \frac{dy}{dx} \ln 3 = -3$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = \frac{4}{\ln 3} \quad (2)$$

(b) Find the equation of the tangent to this curve at the point where $x = 1$
(Leave answer exact)

$$x=1 \Rightarrow 3^{-y} = e^{-3} \Rightarrow -y\ln 3 = -3 \Rightarrow y = \frac{3}{\ln 3} \quad (2)$$

(4) equation: $$y - \frac{3}{\ln 3} = \frac{4}{\ln 3}(x-1) \quad (1)$$

$$\Rightarrow y\ln 3 - 3 = 4x - 4 \Rightarrow 4x - y\ln 3 = 1 \quad (1)$$

(c) Find: $\frac{d^2y}{dx^2}$ at $x = 1$ (Leave answer exact)

from $(*)$ $$6x\ln x + \frac{1}{x} 3x^2 + 2x - \frac{d^2y}{dx^2} \ln 3 = 0 \quad (2)$$

(4) $$x=1 \Rightarrow 0 + 3 + 2 - \frac{d^2y}{dx^2} \ln 3 = 0$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{5}{\ln 3} \quad (2)$$

13. Given: $f(x) = x^2 \ln x$

(a) State the restrictions on x for this function and all intercepts.

(2) $x > 0$ ① $(1, 0)$ ①

(b) i) Find the vertical asymptotes of f. Justify your answer.

(3) $x \to 0^+ \quad \underset{0^+ \quad -\infty}{x^2 \ln x} = \dfrac{\ln x \to -\infty}{\frac{1}{x^2} \to +\infty} = \dfrac{\frac{1}{x}}{\frac{-2}{x^3}} = \dfrac{x^2}{-2} \to 0^-$ ②

∴ no vertical asymptote ① $(0,0)$ is a missing point.

ii) Find the horizontal asymptotes of f. Justify your answer.

(2) $x \to +\infty \quad x^2 \ln x \to +\infty$. ①

∴ no horizontal asymptote ①

(b) i) Show that $f'(x) = x(2\ln x + 1)$

(2) $f'(x) = 2x \ln x + \frac{1}{x} x^2$ ②

$= x(2\ln x + 1)$

ii) State where f is increasing and decreasing and find all relative extrema.

(3) f' is not defined for $x \leq 0$

$f' = 0 \Rightarrow 2\ln x + 1 = 0 \Rightarrow \ln x = \frac{-1}{2} \Rightarrow x = e^{-1/2} = \frac{1}{\sqrt{e}}$

$\approx .6$

$0 \quad f'(\frac{1}{2}) < 0 \quad \frac{1}{\sqrt{e}} \quad f'(1) > 0$

dec ① inc ①

min $\left(\frac{1}{\sqrt{e}}, \frac{-1}{2e}\right)$ ① $-.2$

13. (c) i) Show $f''(x) = 2\ln x + 3$

$f''(x) = 2\ln x + 1 + \frac{2}{x}x = 2\ln x + 3$ (2)

(2)

ii) State where f is concave upward and downward and find all inflection points.

f'' is not defined when $x \le 0$

$f'' = 0 \Rightarrow 2\ln x + 3 = 0 \Rightarrow \ln x = -3/2 \Rightarrow x = e^{-3/2} \approx .22$.

(3)

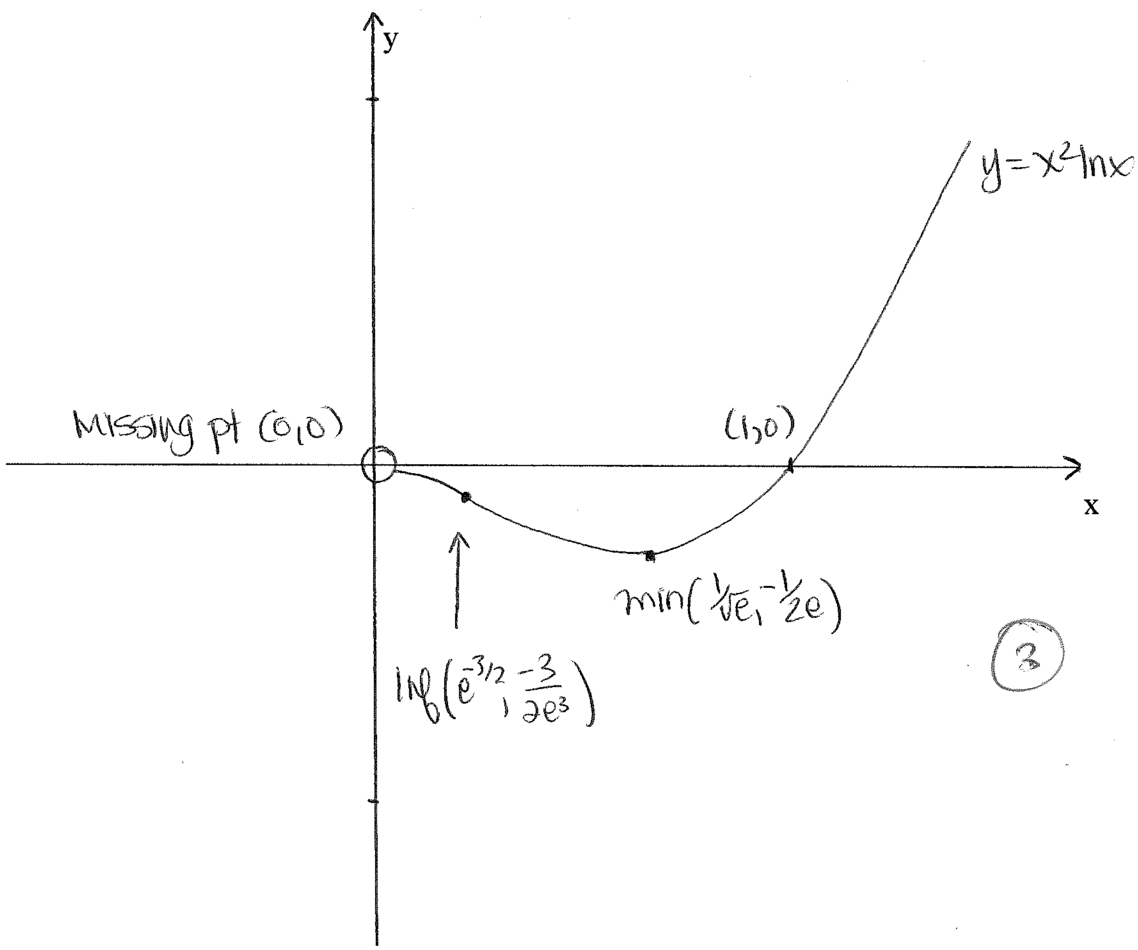
0 $f''(.1) < 0$ $e^{-3/2}$ $f''(1) > 0$

con. down (1) con. up (1)

Inf. $(e^{-3/2}, \frac{-3}{2e^3})$ $= -.07$. (1)

(d) Graph the function clearly on the axes below:

(3)



14. The demand equation for a product is given by: $p = -2q + 200$ and $q = \frac{16m}{m+2}$.

where m is the number of employees needed to produce q units of product.

The average cost function is given by: $\bar{c} = 2q + 40 + \frac{200}{q}$

(a) Find the marginal revenue when $q = 8$

$r = pq = -2q^2 + 200q$. ①

$MR = \frac{dr}{dq} = -4q + 200$

(2)

$MR\Big|_{q=8} = 168$ ①

(b) Find the marginal revenue product when $q = 8$

$MRP = \frac{dr}{dm} = \frac{dr}{dq} \cdot \frac{dq}{dm} = (-4q+200)\left(\frac{16(m+2) - 16m}{(m+2)^2}\right)$ ②

(4)

$MRP\Big|_{\substack{q=8 \\ m=2}} = 168\left(\frac{32}{16}\right) = 336$ ①

ie: $q = 8 \Rightarrow \frac{16m}{m+2} = 8$

$\Rightarrow 16m = 8m + 16$

$\Rightarrow 8m = 16$ ①

$\Rightarrow m = 2$.

(c) Find the value of q which maximizes profit

Profit = Revenue − Cost

$= -2q^2 + 200q - q\left(2q + 40 + \frac{200}{q}\right)$ ②

$P = -4q^2 + 160q - 200$ ①

(6)

$\frac{dP}{dq} = -8q + 160 = 0 \Rightarrow q = 20$ ②

$\frac{d^2P}{dq^2} = -8 < 0$ for all $q > 0 \Rightarrow$ abs. max profit when $q = 20$.

①