MAT 133Y1Y TERM TEST #2

THURSDAY, JULY 5, 2012   7:10 - 9:10 PM

FAMILY NAME: ______________________

GIVEN NAMES: ______________________ Solutions

STUDENT NUMBER: ______________________

SIGNATURE: ______________________

Aids Allowed: Calculator with empty memory, to be supplied by the student. Absolutely no graphing calculators allowed.
Instructions: This test has 10 multiple choice questions worth 4 marks each and 4 written answer questions. For each multiple choice question, you may do your rough work in the test booklet, but you must record your answer by circling one of the letters A, B, C, D or E which appear on the front page of the test.
A multiple choice question left blank, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the written answer solutions, present your solutions in the spaces provided. Use the back of the question pages for your rough work.

ANSWERS FOR MULTIPLE CHOICE **Circle the correct answer**					
1.	**A**	**B**	**C**	**D**	**E**
2.	**A**	**B**	**C**	**D**	**E**
3.	**A**	**B**	**C**	**D**	**E**
4.	**A**	**B**	**C**	**D**	**E**
5.	**A**	**B**	**C**	**D**	**E**
6.	**A**	**B**	**C**	**D**	**E**
7.	**A**	**B**	**C**	**D**	**E**
8.	**A**	**B**	**C**	**D**	**E**
9.	**A**	**B**	**C**	**D**	**E**
10.	**A**	**B**	**C**	**D**	**E**

GRADER'S REPORT	
Multiple Choice	/ 40
Question 11	/ 9
Question 12	/ 16
Question 13	/ 23
Question 14	/ 12
TOTAL	/100

1. If $f(x) = \begin{cases} 3^{(1/x)} + 1 & \text{if } x < 0 \\ a & \text{if } x = 0 \\ 2x - b & \text{if } x > 0 \end{cases}$ is continuous at $x=0$, then

A $a = 2$ & $b = -2$

B $a = 1$ & $b = -1$

C $a = 3$ & $b = -3$

D $a = 4$ & $b = -4$

E There are no values of a and b that make f continuous at x = 0

① $f(0) = a$

② $\lim_{x \to 0^+} 2x - b = -b$

$\lim_{x \to 0^-} 3^{\frac{1}{x}} + 1 = 1$

$\therefore a = 1 = -b$

$\Rightarrow a = 1$

$b = -1$

2. $\lim_{x \to 1} \dfrac{e^x - ex}{(x-1)^2} = \lim_{x \to 1} \dfrac{e^x - e}{2(x-1)} = \lim_{x \to 1} \dfrac{e^x}{2} = \dfrac{e}{2}.$

A 0

B 2e

C e

D e /2

E $+\infty$

3. The solution to the equality: $\dfrac{x^2 - 5x + 6}{\ln x} \leq 0$ is given by:

ie $x > 0$

$$\frac{(x-3)(x-2)}{\ln x} \leq 0$$

A $0 < x < 1$ or $2 \leq x \leq 3$

B $1 < x \leq 2$ or $x \geq 3$

C $x < 1$ or $2 \leq x \leq 3$

D $x \leq 2$ or $x \geq 3$

E $x < 0$ or $2 \leq x \leq 3$

0 1 2 3

− + − +

4. $\lim\limits_{x \to +\infty} \underbrace{(x + e^x)}_{\to +\infty}^{\overset{\to 0}{\frac{1}{x}}} = e^1 = e$

A 0

B 1

C e

D $\sqrt{e}$

E $+\infty$

ie: $\lim\limits_{x \to +\infty} \ln(x+e^x)^{\frac{1}{x}} = \lim\limits_{x \to +\infty} \dfrac{\ln(x+e^x) \to +\infty}{x \to +\infty}$

$= \lim\limits_{x \to +\infty} \dfrac{\frac{1}{x+e^x}(1+e^x)}{1}$

$= \lim\limits_{x \to +\infty} \dfrac{e^x \to +\infty}{1+e^x \to +\infty}$

$= \lim\limits_{x \to +\infty} \dfrac{e^x}{e^x}$

$= \lim\limits_{x \to +\infty} (1) = 1$

5. The function $f(x) = x(3 - x^2)$ on the interval $[-2, 2]$ has:

A an absolute minimum at $x = 2$ only and an absolute maximum at $x = -2$ only

B an absolute minimum at $x = -1, 2$ and an absolute maximum at $x = -2, 1$

C an absolute minimum at $x = -1$ only and an absolute maximum at $x = 1$ only

D an absolute minimum at $x = 1, -2$ and an absolute maximum at $x = 2, -1$

E an absolute minimum at $x = 2, -2$ and an absolute maximum at $x = 1, -1$ and a relative minimum at $x = 0$

$$f'(x) = 3 - x^2 - 2x^2 = 3 - 3x^2 = 3(1 - x^2)$$

$$f'(x) = 0 \Rightarrow x = \pm 1$$

$f(-2) = 2$ $\quad$ $f(1) = 2$

$f(-1) = -2$ $\quad$ $f(2) = -2$.

6. If $f(x) = \sqrt{1 + \sqrt{x}}$ then the relative rate of change of f with respect to x at $x = 9$ is:

A 1/48

B 1/24

C 1/8

D 1/4

E 1/6

$$f'(x) = \frac{1}{2}(1 + x^{1/2})^{-1/2}\left(\frac{1}{2}x^{-1/2}\right)$$

$$= \frac{1}{4}\frac{1}{\sqrt{x}}\frac{1}{\sqrt{1+\sqrt{x}}}$$

$$f'(9) = \frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{1}{24}.$$

relative rate of change at $x=9$ $= \dfrac{f'(9)}{f(9)} = \dfrac{\frac{1}{24}}{2} = \dfrac{1}{48}.$

7. The population of a town at time t is given by: $P(t) = 10{,}000\,(2)^t$
The rate at which the population is changing when $t = 4$ is closest to:

A 230,831

B 160,000

C 320,000

D 80,000

E 110,904

$P'(t) = 10{,}000\, 2^t \cdot \ln 2$.

$P'(4) = 10{,}000\, 2^4 \cdot \ln 2$.

$= 160{,}000 \ln 2$

≈ 110904

8. If a country's Savings (S) and National Income (I) are related by:

$$2S^2 + I^2 = 3SI$$

then when $I = 4$ and $S = 2$, the Marginal Propensity to Save is:

A 3/4

B –8/5

C 5/6

D 1/2

E 2

$4S\frac{dS}{dI} + 2I = 3I\frac{dS}{dI} + 3S$.

$\left.\begin{matrix} I=4 \\ S=2 \end{matrix}\right\} \Rightarrow 8\frac{dS}{dI} + 8 = 12\frac{dS}{dI} + 6$.

$2 = 4\frac{dS}{dI}$

$\therefore MPS = \frac{dS}{dI} = \frac{1}{2}$

9. If the demand function for a product is given by: $p = \dfrac{3000}{q+100}$ and $q = 100m - m^2$ is the total number of units produced by "m" employees.

$r = \dfrac{3000q}{q+100}$

The Marginal Revenue Product, when there are 10 employees is:

A 0.24

B 24.79

C 2.4

D 0.3

E 24

$MRP = \dfrac{dr}{dm} = \dfrac{dr}{dq} \cdot \dfrac{dq}{dm}$

$= \dfrac{3000(q+100) - 3000q}{(q+100)^2}(100-2m)$

$= \dfrac{300000}{(q+100)^2}(100-2m)$

$MRP\Big|_{\substack{m=10 \\ q=900}} = \dfrac{300000}{1000000}(80) = 24.$

10. If we use Newton's Method to approximate the x-intercept of $f(x) = e^x + x$ by starting with $x_1 = 0$ then $x_2 =$

A – 0.57

B – 0.56

C – 0.50

D – 0.43

E – 0.42

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 0 - \dfrac{f(0)}{f'(0)}$

$= -\dfrac{1}{2}.$

11. Evaluate the following limits:
(Show all steps clearly)

(a) $\lim_{x \to +\infty} \dfrac{2x^2 + 5x - 3}{x^2 - 9}$ $\begin{matrix} \to +\infty \\ \to +\infty \end{matrix}$

(2) $= \lim_{x \to +\infty} \dfrac{2 + \frac{5}{x} - \frac{3}{x^2}}{1 - \frac{9}{x^2}}$ $\begin{matrix} \frac{5}{x} \to 0, \ \frac{3}{x^2} \to 0 \\ \frac{9}{x^2} \to 0 \end{matrix}$ 1 $= 2.$ 1

(b) $\lim_{x \to 4^-} \dfrac{x^2 - 2x - 8}{|x-4|}$ $\begin{matrix} \to 0 \\ \to 0 \end{matrix}$ $= \lim_{x \to 4^-} \dfrac{(x-4)(x+2)}{-(x-4)}$ 2

(3) $= \lim_{x \to 4^-} -(x+2) = -6$ 1

$(x+4)$

(c) $\lim_{h \to 0} \dfrac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$ $= \lim_{h \to 0} \dfrac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$ $\begin{matrix} \to 0 \\ \to 0 \end{matrix}$ 1

$= \lim_{h \to 0} \dfrac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \times \dfrac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}.$

$= \lim_{h \to 0} \dfrac{x - (x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}$ 1

(4)

$= \lim_{h \to 0} \dfrac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})}$ 1

$= \dfrac{-1}{2x^{3/2}}$ or $-\frac{1}{2}x^{-3/2}.$ 1

12. (a) Given: $f(x) = \dfrac{(x^2+2)^x \cdot 2^{1/x}}{e^{-\sqrt{x}} \cdot x^{3x}}$ Find: $f'(x)$

$\ln f(x) = x\ln(x^2+2) + \frac{1}{x}\ln 2 - (-\sqrt{x}) - 3x\ln x$ 2

$\frac{1}{f(x)} f'(x) = \ln(x^2+2) + \frac{2x}{x^2+2}\cdot x - \frac{1}{x^2}\ln 2 + \frac{1}{2\sqrt{x}} - 3\ln x - \frac{3x}{x}$

(6)

$f'(x) = \dfrac{(x^2+2)^x \cdot 2^{\frac{1}{x}}}{e^{-\sqrt{x}} \cdot x^{3x}}\left[\ln(x^2+2) + \frac{2x^2}{x^2+2} - \frac{\ln 2}{x^2} + \frac{1}{2\sqrt{x}} - 3\ln x - 3\right]$

4

(b) i) Given: $x^2y^3 = 1$ Find: $\frac{d^2y}{dx^2}$ in terms of x and y only and simplify completely.

$2xy^3 + 3y^2\frac{dy}{dx}x^2 = 0$ 2 $\Rightarrow \frac{dy}{dx} = \frac{-2xy^3}{3x^2y^2} = -\frac{2y}{3x}$ 1

(7) $\frac{d^2y}{dx^2} = \dfrac{-2\frac{dy}{dx}(3x) - 3(-2y)}{(3x)^2}$ 2 $= \dfrac{-2\left(\frac{-2y}{3x}\right)(3x) + 6y}{9x^2}$

$= \dfrac{10y}{9x^2}$ 2

ii) Find the equation of the tangent to $x^2y^3 = 1$ at the point (1,1).

$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-2}{3}$ 1 ∴ equation of tangent is:

(3) $y - 1 = \frac{-2}{3}(x-1)$ 2

or $2x + 3y = 5$

13. Given: $f(x) = \dfrac{(2-x)(1+x)}{(1-x)^2} = \dfrac{2+x-x^2}{(1-x)^2}$

(a) Find the intercepts of f $\quad (0,2)\ (2,0)\ (-1,0)$ 2

(2)

(b) Find the horizontal and vertical asymptotes of f. Justify your answer.

f is not defined at $x=1$

If $\left.\begin{array}{l} x\to 1^+ \\ x\to 1^- \end{array}\right\}$ $f(x) = \dfrac{\overbrace{(2-x)(1+x)}^{\to 2}}{\underbrace{(1-x)^2}_{\to 0^+}} \to +\infty$. 1 $\therefore$ $x=1$ is a V.A. 1

(4)

If $x\to+\infty$ $\quad \dfrac{2+x-x^2}{1-2x+x^2} = \dfrac{\frac{2}{x^2}+\frac{1}{x}-1}{\frac{1}{x^2}-\frac{2}{x}+1} \to -1^-$ $\therefore$ $y=-1$ is a H.A. 1

If. $x\to-\infty$ $\quad f(x)\to -1^+$ 1

(c) i) Show that: $f'(x) = \dfrac{5-x}{(1-x)^3}$

$f'(x) = (1-2x)(1-x)^{-2} + 2(1-x)^{-3}(2+x-x^2)$ 2

(4) $\quad = (1-x)^{-3}\left[(1-2x)(1-x) + 4 + 2x - 2x^2\right]$

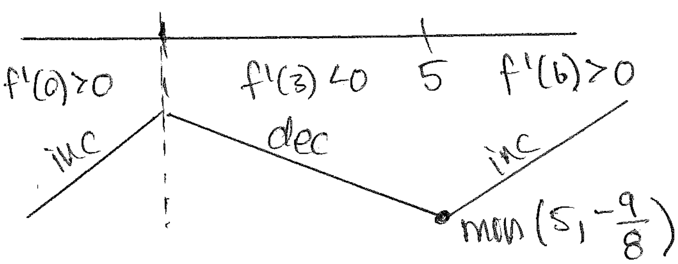
$= (1-x)^{-3}(5-x)$ 2

ii) State where f is increasing, decreasing and all maximum and minimum points.

f' is not defined at $x=1$

$f'(x)=0 \Rightarrow x=5$

(3)



3

13. (e) i) Show that: $f''(x) = \dfrac{2(7-x)}{(1-x)^4}$

(3)

$$f''(x) = (-1)(1-x)^{-3} + 3(1-x)^{-4}(5-x)$$
$$= (1-x)^{-4}\left[-1+x+15-3x\right]$$
$$= (14-2x)(1-x)^{-4} = 2(7-x)(1-x)^{-4}$$

3

ii) State where f is concave upward and concave downward and all inflection points.

(3)

$f''(x)=0 \Rightarrow x=7$ $\quad f''$ is not defined at $x=1$

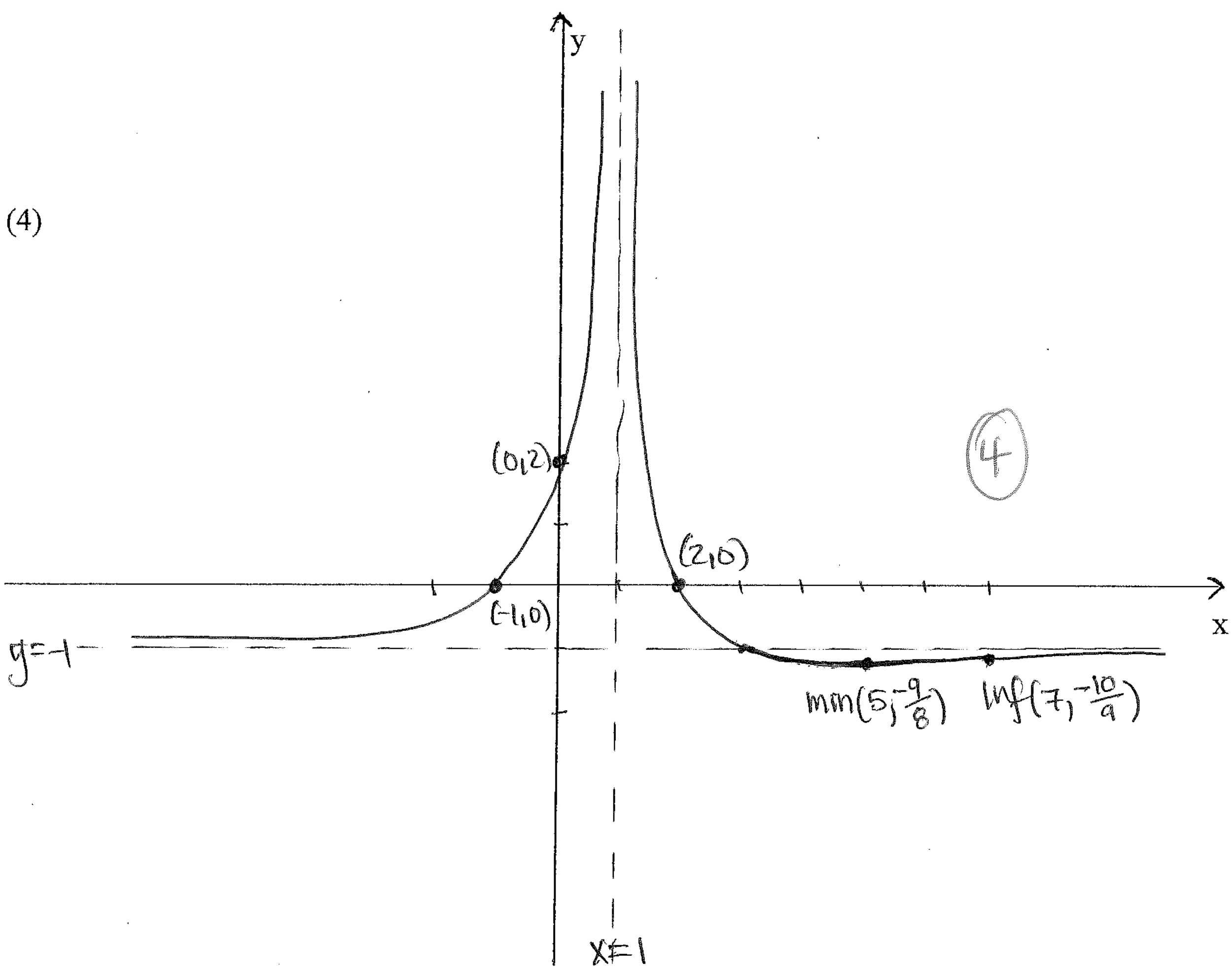
$f''(0)>0$ 1 $\quad f''(5)>0$ 7 $\quad f''(8)<0$

con up $\quad$ con up $\quad$ con down $\quad$ Infl pt $(7, \frac{-10}{9})$

3

(f) Graph the function clearly on the axes below:

(4)



14. The demand curve for a certain product is given by: $p = 360 - q$ and the cost of producing q units is given by: $c = 100 + 3q$ where both p and c are in dollars.

(a) If the Government imposes a tax of \$7 per unit on the product, then how much should be produced in order to maximize the company's profits?

Maximize: Profits = Revenue - cost (1)

$$P = pq - c - 7q$$

$$= 360q - q^2 - 100 - 3q - 7q \quad (2)$$

$$= 350q - q^2 - 100 \quad (1)$$

(8)

$$\frac{dP}{dq} = 350 - 2q = 0 \Rightarrow q = 175 \quad (2)$$

$$\frac{d^2P}{dq^2} = -2 < 0 \text{ for all } q \Rightarrow \text{abs max when } q = 175 \quad (2)$$

$\therefore$ they should produce 175 units in order to maximize profits

(b) Determine the point elasticity of demand for the above product when 100 units are produced. Is the demand for this product elastic or inelastic?

$$\eta = \frac{\frac{p}{q}}{\frac{dp}{dq}} = \frac{\frac{360-q}{q}}{-1} = \frac{q-360}{q} \quad (2)$$

(4)

$$\eta\Big|_{q=100} = \frac{100-360}{100} = \frac{-260}{100} = -2.6 \quad (1)$$

$\therefore$ demand is elastic (1)