MAT 133Y1Y TERM TEST 2

Thursday, 4 July, 2013, 7:10 pm – 9:10 pm

Code 1

FAMILY NAME ______ Solutions

GIVEN NAME(S) ______

STUDENT NO. ______

SIGNATURE ______

GRADER'S REPORT	
Question	Mark
MC/40	
B1/15	
B2/15	
B3/15	
B4/15	
TOTAL	

NOTE:

1. **Aids Allowed:** Calculator with empty memory, to be supplied by student. Absolutely no graphing calculators allowed.
2. **Instructions:** Fill in the information on this page and ensure that the test contains 12 pages.
3. This test has 10 multiple choice questions worth 4 marks each and 4 written-answer questions worth 15 marks each.

   For the **multiple choice questions** indicate your answers by circling the appropriate letters (**A**, **B**, **C**, **D**, or **E**) on **this page (page 1)**. A multiple choice question left blank on **this page**, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the **written-answer questions**, present your solutions in the spaces provided. Use the multiple choice question pages or the back of any of the pages for rough work, for any of the questions.

ANSWER BOX FOR PART A

Circle the correct answer.

1.	A	B	C	D	E
2.	A	B	C	D	E
3.	A	B	C	D	E
4.	A	B	C	D	E
5.	A	B	C	D	E
6.	A	B	C	D	E
7.	A	B	C	D	E
8.	A	B	C	D	E
9.	A	B	C	D	E
10.	A	B	C	D	E

PART A. Multiple Choice

1. *[4 marks]*

$$\lim_{x\to -\infty} \frac{5(x+1)(x+2)(x+3)}{x^2+x+1} =$$

$$\lim_{x\to -\infty} \frac{5(1+x^{-1})(1+2x^{-1})(x+3)}{1+x^{-1}+x^{-2}}$$

A $+\infty$

B 5

$$= \lim_{x\to -\infty} \frac{5(x+3)}{1}$$

C $-\infty$

D -5

E 3

$$= -\infty$$

2. *[4 marks]*

$\dfrac{|x^3+1|}{x} < 0$ precisely when

$|x^3+1| > 0$ for all x

except $x = -1$

A $x < -1$ or $0 < x < 1$

B $-1 < x < 1$

C $-1 < x < 0$ or $x > 1$

D $x < -1$ or $x > 1$

E $x < 0$ and $x \neq -1$

3. *[4 marks]*

The line $y = ax + b$, which is tangent at $(x, y) = (1, 5)$ to the graph of $f(x) = \dfrac{10 + \ln x}{1 + x}$, has y-intercept $b =$

A 6

B 7

C 5

D 4

E 8

$$f'(x) = \frac{x^{-1}(1+x) - (10 + \ln x)}{(1+x)^2}$$

$$f'(1) = \frac{2 - (10 + \ln 1)}{2^2} = -2 \text{ and}$$

the tangent line has equation

$$y = -2(x-1) + 5 = -2x + 7$$

4. *[4 marks]*

If a revenue function is given, in dollars, by

$$r(q) = (300 - q)q \quad \text{so } r'(q) = 300 - 2q$$

and a production function is given by

$$q = 100\left(1 - \frac{1}{m}\right) \quad \text{so } q'(m) = \frac{100}{m^2}$$

where m is the number of employees, and q is the quantity produced and sold, then, when $m = 5$, the marginal revenue product, $\dfrac{dr}{dm} =$

A \$570 per employee

B \$540 per employee

C \$550 per employee

D \$580 per employee

E \$560 per employee

when $m = 5$, $q'(5) = \frac{100}{5^2} = 4$,

$q(5) = 80$, and $r'(80) = 300 - 160 = 140$.

So $\frac{dr}{dm}(5) = \frac{dr}{dq}(80) \cdot \frac{dq}{dm}(5) = 140 \cdot 4$

5. *[4 marks]*

If $y = (x+1)^{-1}(x+2)^{-2}(x+3)^{-3}$, then $y'(0) =$

A $-\frac{1}{27}$

B $-\frac{1}{12}$

C $-\frac{1}{54}$

D $-\frac{1}{36}$

E $-\frac{1}{18}$

Logarithmic differentiation:

$\ln y = -\ln(x+1) - 2\ln(x+2) - 3\ln(x+3)$

so that $\frac{y'}{y} = -\frac{1}{x+1} - \frac{2}{x+2} - \frac{3}{x+3}$

At $x=0$: $\frac{y'(0)}{y(0)} = -\frac{1}{1} - \frac{2}{2} - \frac{3}{3} = -3$

But $y(0) = 1^{-1}2^{-2}3^{-3} = \frac{1}{108}$ so $y'(0) = -3 \cdot \frac{1}{108}$

6. *[4 marks]*

If a product has demand function $p = e^{-2q}$, then its point elasticity of demand when $q = 3$ is

A $-\frac{1}{3}$

B $-\frac{1}{6}$

C $-\frac{1}{12}$

D $-\frac{1}{2}$

E $-\frac{1}{4}$

$\frac{dp}{dq} = -2e^{-2q}$

When $q = 3$,

$$\frac{\left(\frac{p}{q}\right)}{\left(\frac{dp}{dq}\right)} = \frac{\left(\frac{e^{-2q}}{q}\right)}{-2e^{-2q}} = -\frac{1}{2q} = -\frac{1}{6}$$

7. *[4 marks]*

If Newton's method is used to estimate a root of the equation $e^x = 4x$ and the initial estimate is $x_1 = 2$, then the next estimate, x_2, is closest to

A 0.36

B 2.18

C 1.37

D 2.15

E 2.25

Let $f(x) = e^x - 4x$ so $f'(x) = e^x - 4$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{e^2 - 8}{e^2 - 4}$$

$$\approx 2.1802696\ldots$$

8. *[4 marks]*

indeterminate, type $\frac{0}{0}$

$$\lim_{x \to 1} \frac{e^x - ex}{(x-1)^2} = \lim_{x \to 1} \frac{e^x - e}{2(x-1)}$$

$$= \lim_{x \to 1} \frac{e^x}{2}$$

by L'Hôpital's rule

A $\frac{1}{2}$

B 2

C e

D $\frac{1}{2}e$

E ∞

9. *[4 marks]*

Let m denote the absolute minimum value of $f(x) = x^3 + 3x^2 - 9x$ on the interval $[-2, 2]$ and let M denote the absolute maximum value of the same function on the same interval. Then $m + M =$

A 22

B −3

C 17

D 24

E 29

$f'(x) = 3x^2 + 6x - 9 = 3(x+3)(x-1)$

so critical points are -2, 1, and 2 (but not -3; $-3 < -2$).

$f(-2) = \underline{22}$ $f(1) = \underline{-5}$ $f(2) = 2$

maximum value minimum value

10. *[4 marks]*

Let c be a fixed real number. The function $f(x) = x^4 + cx^2$ has exactly two inflection points

A if and only if $c > 0$

B for any c

C for no value of c

D if and only if $c < 0$

E if and only if $c \leq 0$

$f''(x) = 12x^2 + 2c$
$= 2(6x^2 + c)$

If $c \geq 0$, $f''(x) \geq 0$ for all x.

If $c < 0$, inflection points are $x = -\sqrt{\frac{-c}{6}}$ and $x = \sqrt{\frac{-c}{6}}$

PART B. Written-Answer Questions
SHOW YOUR WORK.

B1.(a) *[7 marks]*

Evaluate $\displaystyle\lim_{x\to 1}\frac{|x^2-1|}{|x|-1}$ or show that the limit does not exist. **Show your work.**

For $x>0$, $|x|=x$ and the limit is

$$\lim_{x\to 1}\frac{|x^2-1|}{x-1}$$

But $\displaystyle\lim_{x\to 1^+}\frac{|x^2-1|}{x-1}=\lim_{x\to 1^+}\frac{x^2-1}{x-1}=\lim_{x\to 1^+}x+1=2$

while $\displaystyle\lim_{x\to 1^-}\frac{|x^2-1|}{x-1}=\lim_{x\to 1^-}\frac{1-x^2}{x-1}=\lim_{x\to 1^-}-(x+1)=-2$

Since the left and right limits are unequal, the given two-sided limit does not exist.

B1.(b) *[8 marks]*

For which values of a and b is the function

$$f(x) = \begin{cases} \dfrac{1}{\ln(1-x)} & \text{if} \quad x < 1 \\ ax + b & \text{if} \quad 1 \le x \le 2 \\ \dfrac{\sqrt{x^2+5}-3}{x-2} & \text{if} \quad x > 2 \end{cases}$$

continuous at all x? **Show your work.**

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1}{\ln(1-x)} = 0$ because

as $x \to 1^-$, $1-x \to 0^+$, and $\ln(1-x) \to -\infty$.

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{\sqrt{x^2+5}-3}{x-2} = \lim_{x \to 2^+} \frac{x^2+5-9}{(x-2)(\sqrt{x^2+5}+3)}$$

$$= \lim_{x \to 2^+} \frac{x+2}{\sqrt{x^2+5}+3} = \frac{2}{3}$$

So f is continuous at all x only if $f(1) = 0$ and $f(2) = \frac{2}{3}$, that is,

$$a + b = 0$$
$$2a + b = \frac{2}{3}$$

This system has solution $a = \frac{2}{3}$, $b = -\frac{2}{3}$

B2. *[15 marks]*

Suppose $y(x)$ satisfies $5x^2 + 4xy + y^2 = 2$.

B2.(a) *[8 marks]*

Find $\dfrac{dy}{dx}$ in terms of x and y.

$$10x + 4y + 4x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$(2x+y)\frac{dy}{dx} = -(5x+2y) \qquad (\star)$$

$$\frac{dy}{dx} = -\frac{5x+2y}{2x+y}$$

B2.(b) *[7 marks]*

Find $\dfrac{d^2y}{dx^2}$ when $(x, y) = (1, -1)$.

From $(\star)$, $(2x+y)\dfrac{d^2y}{dx^2} + \left(2+\dfrac{dy}{dx}\right)\dfrac{dy}{dx} = -5 - 2\dfrac{dy}{dx}$

When $x=1$ and $y=-1$, $\dfrac{dy}{dx} = -3$ (from B2.(a)),

and $(2\cdot1-1)\dfrac{d^2y}{dx^2} + (2-3)(-3) = -5-2(-3)$,

$\dfrac{d^2y}{dx^2} + 3 = -5+6$, and $\dfrac{d^2y}{dx^2} = -2$

B3. *[15 marks]*

Let $f(x) = \dfrac{x^2+4}{(x-2)^2}$. Note that $f'(x) = \dfrac{-4x-8}{(x-2)^3}$ and $f''(x) = \dfrac{8x+32}{(x-2)^4}$.

B3.(a) *[2 marks]*

Find all vertical and horizontal asymptotes of $y = f(x)$.

vertical asymptote: $x=2$

horizontal asymptote: $y=1$

because $\lim\limits_{x\to\pm\infty} \dfrac{x^2+4}{(x-2)^2} = \lim\limits_{x\to\pm\infty} \dfrac{1+\frac{4}{x^2}}{\left(1-\frac{2}{x}\right)^2} = 1$

B3.(b) *[3 marks]*

Find all intervals where f is increasing, intervals where f is decreasing, relative minima, and relative maxima.

f increases on $(-2,2)$, decreases on $(-\infty,-2)$ and on $(2,\infty)$.

f takes a relative minimum at $x=-2$.

B3.(c) *[3 marks]*

Find all intervals where f is concave up, intervals where f is concave down, and inflection points.

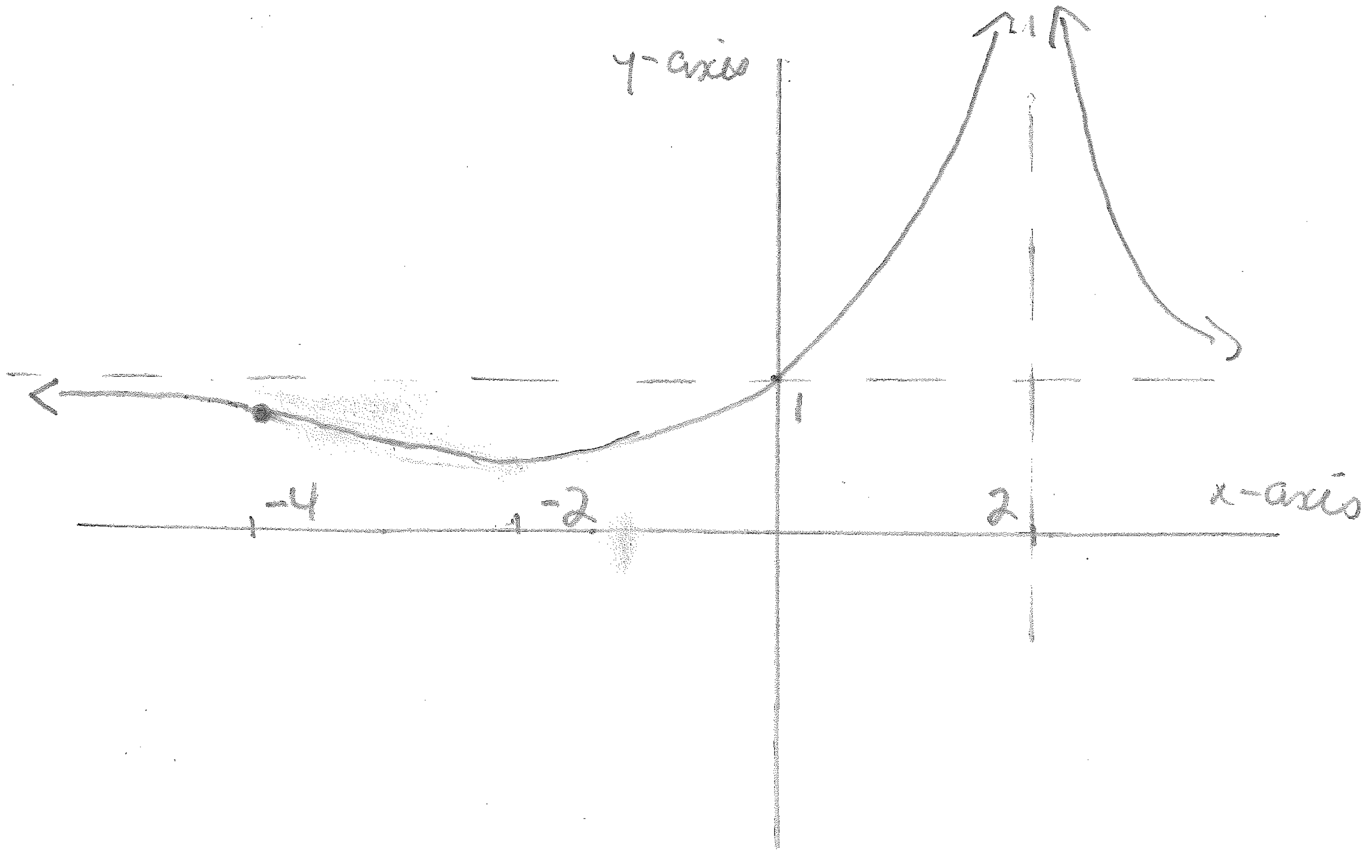
f is concave up on $(-4,2)$ and on $(2,\infty)$, concave down on $(-\infty,-4)$.

f has an inflection point at $x=-4$.

B3.(d) *[7 marks]*

Sketch the graph of f.

Recall that $f(x) = \dfrac{x^2+4}{(x-2)^2}$, $f'(x) = \dfrac{-4x-8}{(x-2)^3}$, and $f''(x) = \dfrac{8x+32}{(x-2)^4}$.



B4. *[15 marks]*

It costs the Acme Company $25q$ dollars to produce q tons of its product; to sell those q tons with no unsatisfied demand, Acme must set its selling price at $p = 250 - 3q^2$ dollars per ton. How many dollars per ton should Acme charge to maximize its profit? Remember to verify that your solution actually maximizes profit.

revenue $= qp = 250q - 3q^3$

profit = revenue − cost

$= 250q - 3q^3 - 25q$

$= 225q - 3q^3$

Let $f(q) = 225q - 3q^3$ (profit function)

To maximize f, $f'(q) = 225 - 9q^2$, so $q = 5$ is critical.

$f''(q) = -18q < 0$ at $q = 5$, so $q = 5$ actually maximizes f.

Acme should charge

$p(5) = 250 - 3 \cdot 5^2 = 175$ dollars per ton to maximize profit.