MAT 133Y1Y TERM TEST 2

Thursday, 3 July, 2014, 6:10 pm – 8:00 pm

Code 1

FAMILY NAME Solutions

GIVEN NAME(S) ______________________

STUDENT NO. ______________________

SIGNATURE ______________________

GRADER'S REPORT	
Question	Mark
MC/40	
B1/15	
B2/15	
B3/15	
B4/15	
TOTAL	

NOTE:

1. **Aids Allowed:** Calculator with empty memory, to be supplied by student. Absolutely no graphing calculators allowed.
2. **Instructions:** Fill in the information on this page and ensure that the test contains 10 pages.
3. This test has 10 multiple choice questions worth 4 marks each and 4 written-answer questions worth 15 marks each.

   For the **multiple choice questions** indicate your answers by circling the appropriate letters (**A**, **B**, **C**, **D**, or **E**) on **this page (page 1)**. A multiple choice question left blank on **this page**, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the **written-answer questions**, present your solutions in the spaces provided. Use the multiple choice question pages or the back of any of the pages for rough work, for any of the questions.

ANSWER BOX FOR PART A

Circle the correct answer.

1.	A	B	C	D	E
2.	A	B	C	D	E
3.	A	B	C	D	E
4.	A	B	C	D	E
5.	A	B	C	D	E
6.	A	B	C	D	E
7.	A	B	C	D	E
8.	A	B	C	D	E
9.	A	B	C	D	E
10.	A	B	C	D	E

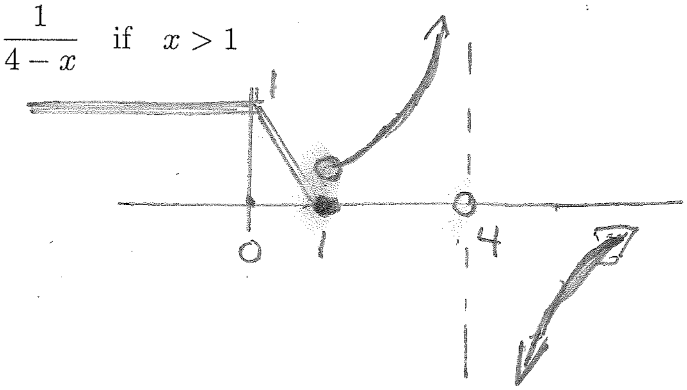
PART A. Multiple Choice

1. *[4 marks]*

If f is defined for all real x except $x = 4$ by

$$f(x) = \begin{cases} 1 & \text{if } x \le 0 \\ 1 - x & \text{if } 0 < x \le 1 \\ \dfrac{1}{4-x} & \text{if } x > 1 \end{cases}$$

then f is not continuous at



A $x = 1$ only

B $x = 0$ only

C $x = 0$ and $x = 4$ only

D $x = 0$, $x = 1$, and $x = 4$ only

E $x = 1$ and $x = 4$ only ← f is continuous at $x=0$: $\lim_{x\to 0^-} f(x) = \lim_{x \to 0^-} 1$, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1-x$, and $f(1)$ all equal 1.

$f(1) = 1-1 = 0$ but $\lim_{x\to 1^+} f(x) = \frac{1}{4-1} \neq 0$

f is not even defined at $x=4$.

2. *[4 marks]*

If $f(x) = \begin{cases} 1 + 3^{\left(\frac{1}{x+1}\right)} & \text{if } x < -1 \\ b & \text{if } x = -1 \\ \dfrac{a}{x+3} & \text{if } x > -1 \end{cases}$ then which of the following is true?

A There are no values of a and b that make f continuous at all real x.

B If $a = 4$ and $b = 2$, then f is continuous at all real x.

C If $a = 2$ and $b = 1$, then f is continuous at all real x, except at $x = -3$.

D If $a = 2$ and $b = 1$, then f is continuous at all real x.

E If $a = 4$ and $b = 2$, then f is continuous at all real x, except at $x = -3$.

For continuity at all real x, $b = \lim_{x\to -1^-} 1 + 3^{\left(\frac{1}{x+1}\right)} = 1$.

Hence $b = \lim_{x \to -1^+} \frac{a}{x+3} = \frac{a}{2}$.

3. *[4 marks]*

The solutions of the inequality $\frac{1+x}{x(1-x)} \geq 0$ are ← Call this $f(x)$.

A $x \leq -1$ and $x > 1$ only

B $-1 \leq x < 0$ and $0 < x < 1$ only

(C) $x \leq -1$ and $0 < x < 1$ only

D $x < -1$ and $0 \leq x \leq 1$ only

E $-1 \leq x < 0$ and $x > 1$ only

-1 0 1 x-axis

$f>0$ $f(-1)=0$ $f<0$ $f(0)$ not defined $f>0$ $f(1)$ not defined $f<0$

4. *[4 marks]*

If $f(8) = 2$ and $f'(8) = 3$, then $\lim_{h \to 0} \frac{\ln(f(8+h)) - \ln 2}{h} =$

A $\ln 3$

B 6

C $\frac{\ln 3}{2}$

(D) $\frac{3}{2}$

E 3

Let $g(x) = \ln(f(x))$ so that

$g(8+h) = \ln(f(8+h))$,

$g(8) = \ln(f(8)) = \ln 2$, and

$g'(8) = \lim_{h \to 0} \frac{g(8+h) - g(8)}{h}$ by

the definition of $g'(8)$. Thus $g'(8)$ equals the limit in the question. But $g'(8) = \frac{f'(8)}{f(8)}$ by the chain rule.

5. *[4 marks]*

The y-intercept (that is, b), of the line $y = ax + b$ which is tangent at $(1, e)$ to the graph of $f(x) = x^e e^x$, equals

(A) $-e^2$

B e^2

C $1 - e^2 - e^3$

D e

E $1 - e$

$f'(x) = (e x^{e-1}) \cdot e^x + x^e \cdot e^x$ by the product rule.

$f'(1) = e \cdot e + e = e^2 + e$

Since $(0, b)$ and $(1, e)$ both lie on the tangent line, $\frac{e-b}{1-0} = f'(1) = e^2 + e$ and $e - b = e^2 + e$ so that $b = e - (e^2 + e)$.

6. *[4 marks]*

If $f(x) = \frac{(x+1)^2}{x(x+2)}$ then $f'(1) =$

(A) $-\frac{4}{9}$

B $\frac{4}{3}$

C 0

D -1

E $\frac{4}{9}$

Quotient rule (with $f(x) = \frac{(x+1)^2}{x^2+2x}$):

$f'(x) = \frac{2(x+1)(x^2+2x) - (x+1)^2(2x+2)}{(x^2+2x)^2}$

$f'(1) = \frac{2 \cdot 2 \cdot 3 - 2^2 \cdot 4}{3^2} = -\frac{4}{9}$

Alternate solution by logarithmic differentiation (note that $f(1) = \frac{4}{3}$): $\ln f = 2\ln(x+1) - \ln x - \ln(x+2)$ so that $\frac{f'}{f} = \frac{2}{x+1} - \frac{1}{x} - \frac{1}{x+2}$. At $x = 1$, $f'(1) = \left(\frac{2}{1+1} - \frac{1}{1} - \frac{1}{1+2}\right) f(1)$

$= -\frac{1}{3} \cdot \frac{4}{3}$

7. *[4 marks]*

If $y = 2^x x^{(x^3)}$ then when $x = 1$, $\dfrac{dy}{dx} =$

A $3 + 2\ln 2$

B $1 + 2\ln 2$

C $1 + \ln 2$

D $6 + \ln 2$

(E) $2 + 2\ln 2$

Logarithmic differentiation (when $x=1$, $y=2$):

$\ln y = x\ln 2 + x^3 \ln x$

$\dfrac{y'}{y} = \ln 2 + \dfrac{x^3}{x} + 3x^2 \ln x$

$\dfrac{y'(1)}{y(1)} = (\ln 2) + 1$ so $y'(1) = 2((\ln 2)+1)$

8. *[4 marks]*

If $x_1 \neq 1$ is used as an initial estimate for a solution of $f(x) = 0$, when $f(x) = \dfrac{x}{1-x}$, then Newton's method yields the second estimate $x_2 =$

A $\sqrt{x_1}$

(B) ${x_1}^2$

C $2x_1$

D 0

E $\frac{1}{2}x_1$

$f'(x) = \dfrac{1-x-x(-1)}{(1-x)^2} = \dfrac{1}{(1-x)^2}$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= x_1 - \dfrac{\left(\frac{x_1}{1-x_1}\right)}{\left(\frac{1}{(1-x_1)^2}\right)}$

$= x_1 - x_1(1-x_1)$

9. *[4 marks]*

If $f(x) = \begin{cases} -x^2 - 4x - 3 & \text{if} \quad -3 \le x \le 0 \\ \dfrac{1}{x} & \text{if} \quad 0 < x \le 1 \end{cases}$ then on the interval $[-3, 1]$, f has

A neither an absolute minimum nor an absolute maximum

B no absolute minimum, but has an absolute maximum at $x = -2$

C an absolute minimum at $x = -3$. and an absolute maximum at $x = -2$

(D) an absolute minimum at $x = 0$, but has no absolute maximum

E an absolute minimum at $x = -2$ and an absolute maximum at $x = 0$

As $x \to 0^+$, $f(x) = \frac{1}{x} \to +\infty$ so f cannot have an absolute maximum; only (A) and (D) are possibly correct. Let $p(x) = -x^2 - 4x - 3$ on $[-3, 0]$. $p'(x) = -2x - 4$ and critical points are $x = -3$, $x = -2$, $x = 0$ where $f(x) = p(x) = 0, 1,$ and -3 respectively. But $f(x) > 0$ for $x > 0$, so $f(0) = -3$ is minimal.

10. *[4 marks]*

How many inflection points does the function $f(x) = x^6 - x^4$ have?

A 1

B 0

(C) 2

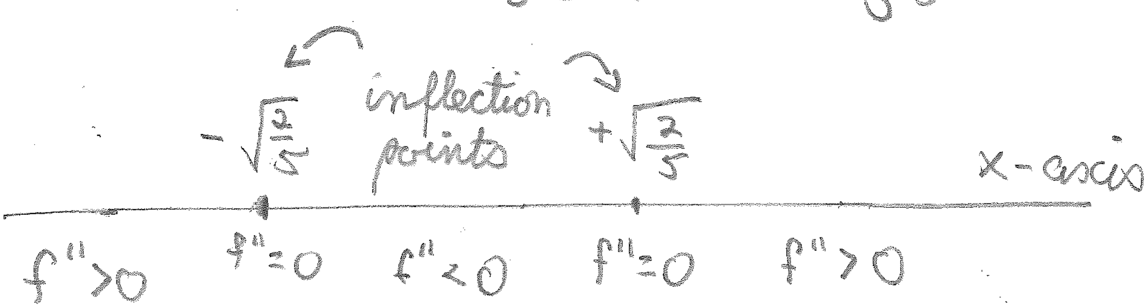
D 4

E 3

$f'(x) = 6x^5 - 4x^3$

$f''(x) = 30x^4 - 12x^2$

$= 30x^2\left(x^2 - \frac{2}{5}\right)$



PART B. Written-Answer Questions

B1. *[15 marks]*

In each part-question below, find the limit, or show that it does not exist.

Show your work.

B1.(a) *[5 marks]*

$$\lim_{x\to 3} \frac{\sqrt{x^2+7}-4}{x-3}$$ is indeterminate of type $\frac{0}{0}$ so by L'Hôpital's rule equals $\lim_{x\to 3} \frac{\frac{1}{2}\cdot\frac{2x}{\sqrt{x^2+7}}}{1} = \boxed{\frac{3}{4}}$.

(Alternatively, an algebraic rearrangement yields $\lim_{x\to 3} \frac{(x^2+7)-4^2}{(x-3)(\sqrt{x^2+7}+4)} = \lim_{x\to 3} \frac{x+3}{\sqrt{x^2+7}+4} = \boxed{\frac{3}{4}}$.)

B1.(b) *[5 marks]*

$$\lim_{x\to -8^-} \frac{x+8}{|64-x^2|}$$

$= \lim_{x\to -8^-} \frac{x+8}{x^2-64}$ (since $x<-8$ implies $x^2>64$ and $64-x^2<0$)

$= \lim_{x\to -8^-} \frac{1}{x-8} = \boxed{-\frac{1}{16}}$ (Again, L'Hôpital's rule provides an alternate method of solution.)

B1.(c) *[5 marks]*

$$\lim_{x\to 1} \frac{\ln(2-x)+x-1}{(x-1)^2}$$ is indeterminate of type $\frac{0}{0}$ so by L'Hôpital's rule equals $\lim_{x\to 1} \frac{\frac{-1}{2-x}+1}{2(x-1)} = \lim_{x\to 1} \frac{-1+(2-x)}{2(x-1)(2-x)}$

$= \lim_{x\to 1} \frac{-1}{2(2-x)} = \boxed{-\frac{1}{2}}$

(Alternatively, a second application of L'Hôpital's rule leads to the same result.)

B2. *[15 marks]*

The Eight Ball Pool Hall has 36 pool tables, which it rents by the hour. The management can rent all the tables if they charge \$5.00 per hour (per table) but each increase of \$0.25 in hourly rent causes one table to be vacant. **Show your work.**

B2.(a) *[8 marks]*

What hourly rent (per table) should the Eight Ball Pool Hall charge to maximize its hourly revenue?

Let x denote the number of \$0.25 increases above \$5.00 per hour.

$\text{Revenue}(x) = (36-x)(5+.25x) = -.25x^2+4x+180$

$\frac{d}{dx}(\text{Revenue}) = -.5x+4$ so $x=8$ is critical.

and, because of B2.(b), maximizes revenue.

When $x=8$, hourly rent $= 5+.25\cdot 8 = 7$ (dollars).

B2.(b) *[2 marks]*

Show that your solution to question 6.(a) does indeed maximize the pool hall's hourly revenue. Let $R(x)$ = revenue. Two solutions are given here.

First derivative test: if $x<8$, $R'>0$ and if $x>8$, $R'<0$.

Second derivative test: $R'' = -.5 < 0$ for all x.

B2.(c) *[5 marks]*

The cost of operating the Eight Ball Pool Hall is \$0.50 per hour for each table rented out. What hourly rent (per table) should it charge to maximize its hourly profit?

$\text{Profit}(x) = \text{Revenue}(x) - \text{Cost}(x)$

$= (36-x)(5+.25x) - (36-x)\cdot .5$

$= (36-x)(4.5+.25x) = -.25x^2+4.5x+162$

$\frac{d}{dx}(\text{Profit}) = -.5x+4.5$ so $x=9$ is critical and the rent which maximizes profit is $5+.25\cdot 9 = 7.25$ (dollars per hour)

B3. *[15 marks]*

Suppose $y(x)$ satisfies $x^2 + y^2 = xy + 1$.

B3.(a) *[6 marks]*

Find $\dfrac{dy}{dx}$ in terms of x and y only.

By implicit differentiation:

$$2x + 2y\frac{dy}{dx} = y + x\frac{dy}{dx} \text{ so } (2y-x)\frac{dy}{dx} = y - 2x$$

and $$\frac{dy}{dx} = \frac{y-2x}{2y-x}.$$

B3.(b) *[9 marks]*

Find $\dfrac{d^2y}{dx^2}$ in terms of x and y only.

From the second equality in the solution of B3.(a),

$$(2y-x)\frac{d^2y}{dx^2} + \left(2\frac{dy}{dx} - 1\right)\frac{dy}{dx} = \frac{dy}{dx} - 2 \text{ so that}$$

$$(2y-x)\frac{d^2y}{dx^2} = \left(1 - 2\frac{dy}{dx}\right)\frac{dy}{dx} + \frac{dy}{dx} - 2 = 2\left(\left(1-\frac{dy}{dx}\right)\frac{dy}{dx} - 1\right).$$

Then $$\frac{d^2y}{dx^2} = 2\,\frac{\left(1 - \frac{y-2x}{2y-x}\right)\frac{y-2x}{2y-x} - 1}{2y-x} = 2\,\frac{\big((2y-x)-(y-2x)\big)(y-2x) - (2y-x)^2}{(2y-x)^3}$$

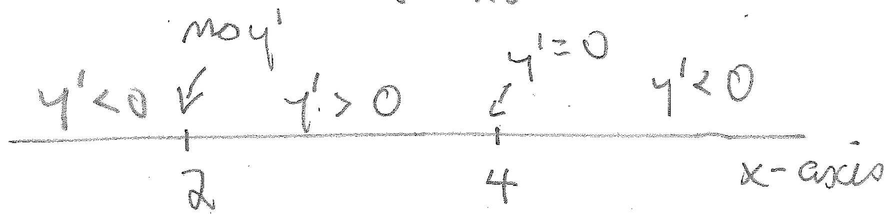
$$= 2\,\frac{(y+x)(y-2x) - (2y-x)^2}{(2y-x)^3} = -6\,\frac{x^2 - xy + y^2}{(2y-x)^3} = -\frac{6}{(2y-x)^3}.$$

Any <u>one</u> of these 5 expressions answers B3.(b).

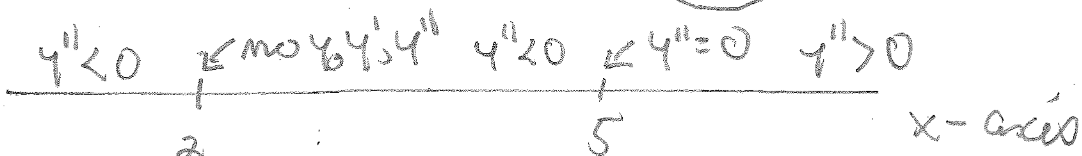
B4. *[15 marks]*

Sketch the graph of $y = \dfrac{4x-12}{(x-2)^2}$. On or near the sketch indicate any horizontal asymptotes, vertical asymptotes, critical points, intervals where y is increasing, intervals where y is decreasing, relative minima, relative maxima, absolute minima, absolute maxima, inflection points, intervals where y is concave up, and intervals where y is concave down. **Note:** $y' = \dfrac{4(4-x)}{(x-2)^3}$ and $y'' = \dfrac{8(x-5)}{(x-2)^4}$.

$\lim_{x\to\pm\infty} y = \lim_{x\to\pm\infty} \dfrac{\frac{4}{x} - \frac{12}{x^2}}{\left(1-\frac{2}{x}\right)^2} = 0$. Horizontal asymptote is $y=0$, vertical asymptote is $x=2$.



Critical point at $x=4$ (but not $x=2$), y increases on $(2,4)$, decreases on $(-\infty,2)$ and on $(4,\infty)$, has a relative maximum at $x=4$ but no relative minimum.



$y(x)$ has an inflection point at $x=5$, is concave up on $(5,\infty)$, concave down on $(-\infty,2)$ and on $(2,5)$ (but not on $(-\infty,5)$).