MAT 133Y1Y TERM TEST 2

Thursday, 2 July, 2015, 6:10 pm – 8:00 pm

Code 1

FAMILY NAME Solutions

GIVEN NAME(S) ______

STUDENT NO. ______

SIGNATURE ______

GRADER'S REPORT	
Question	Mark
MC/40	
B1/15	
B2/15	
B3/15	
B4/15	
TOTAL	

NOTE:

1. **Aids Allowed:** Calculator with empty memory, to be supplied by student. Absolutely no graphing calculators allowed.
2. **Instructions:** Fill in the information on this page and ensure that the test contains 12 pages.
3. This test has 10 multiple choice questions worth 4 marks each and 4 written-answer questions worth 15 marks each.
   For the **multiple choice questions** indicate your answers by circling the appropriate letters (**A**, **B**, **C**, **D**, or **E**) on **this page (page 1)**. A multiple choice question left blank on **this page**, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the **written-answer questions**, present your solutions in the spaces provided. Use the multiple choice question pages or the back of any of the pages for rough work, for any of the questions.

ANSWER BOX FOR PART A
Circle the correct answer.

1.	A	B	C	D	E
2.	A	B	C	D	E
3.	A	B	C	D	E
4.	A	B	C	D	E
5.	A	B	C	D	E
6.	A	B	C	D	E
7.	A	B	C	D	E
8.	A	B	C	D	E
9.	A	B	C	D	E
10.	A	B	C	D	E

PART A. Multiple Choice

1. *[4 marks]*

$$\lim_{x\to 1}\left(\frac{4x}{x^2-1}-\frac{2}{x-1}\right) = \lim_{x\to 1} \frac{4x-2(x+1)}{(x-1)(x+1)}$$

$$= \lim_{x\to 1} \frac{2(x-1)}{(x-1)(x+1)}$$

A $=-2$

B $=1$

C $=2$

D $=0$

E does not exist

2. *[4 marks]*

Let $f(x)=\begin{cases} 2^{\frac{1}{x+1}}+a & \text{if } \ x<-1 \\ \dfrac{x^2-1}{x-1} & \text{if } \ -1\le x<1 \\ b-x & \text{if } \ x\ge 1 \end{cases}$.

If f is continuous at $x=-1$ and $x=1$, then

A $a=-1$ and $b=-1$

B $a=0$ and $b=1$

C $a=0$ and $b=3$

D $a=2$ and $b=3$

E $a=0$ and $b=2$

$f(-1) = \lim_{x\to -1^-} 2^{\frac{1}{x+1}} + a = a$

because as $x\to -1^-$, $x+1\to 0^-$, so $\frac{1}{x+1}\to -\infty$ and $2^{\frac{1}{x+1}}\to 0$.

Also $f(-1) = \frac{(-1)^2-1}{(-1)-1} = 0$

similarly, $f(1) = \lim_{x\to 1^-} \frac{x^2-1}{x-1} = 2$ and $f(1) = b-1$.

3. *[4 marks]*

The equation of the line tangent to the graph of $y = \dfrac{(x^2-3)^3}{\sqrt{4-3x}}$ at the point $(1,-8)$ is

A $y = 16x - 24$

B $y = 12x - 20$

C $2y = 51x - 67$

D $y = -12x + 4$

E $y = -16x + 8$

$\ln y = 3\ln(x^2-3) - \frac{1}{2}\ln(4-3x)$

so that

$\dfrac{y'}{y} = \dfrac{6x}{x^2-3} + \dfrac{3}{2(4-3x)}$

when $x=1$, $y=-8$, and $\dfrac{y'}{-8} = \dfrac{6}{-2} + \dfrac{3}{2} = -\dfrac{3}{2}$

So $y' = 12$ and the line has equation

$y = 12(x-1) - 8$

4. *[4 marks]*

If $y^3 + xy = 2x^2$, then when $x = 1$ and $y = 1$, $\dfrac{dy}{dx} =$

A $\frac{3}{4}$

B 1

C $\frac{1}{2}$

D $\frac{5}{4}$

E $\frac{1}{4}$

Implicit differentiation:

$3y^2\dfrac{dy}{dx} + x\dfrac{dy}{dx} + y = 4x$

If $x=1$, $y=1$,

$3\dfrac{dy}{dx} + \dfrac{dy}{dx} + 1 = 4$

$f(4) = 4^{\sqrt{4}} = 16$

5. *[4 marks]*

If $f(x) = x^{\sqrt{x}}$, then $f'(4) =$

A $8 + 8\ln 4$

B $4 + 4\ln 4$

C $4 + 8\ln 4$

D $8 + 4\ln 4$

E $4 - 8\ln 4$

$\ln f(x) = \sqrt{x} \ln x$

$$\frac{f'(x)}{f(x)} = \frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}$$

$$\frac{f'(4)}{16} = \frac{1}{\sqrt{4}} + \frac{\ln 4}{2\sqrt{4}}$$

$$= \frac{1}{2} + \frac{\ln 4}{4}$$

6. *[4 marks]*

If a product has demand function $p = e^{-2q}$, then its point elasticity of demand when $q = 4$ is

A $-\frac{1}{4}$

B $-\frac{1}{8}$

C $-\frac{1}{16}$

D $-\frac{1}{2}$

E -1

$\frac{dp}{dq} = -2e^{-2q}$ so that

$$\frac{\left(\frac{p}{q}\right)}{\left(\frac{dp}{dq}\right)} = \frac{\left(\frac{e^{-2q}}{q}\right)}{-2e^{-2q}} = -\frac{1}{2q}$$

7. *[4 marks]*

If Newton's method is used to approximate a root of the equation $x^4 - 4x + 1 = 0$ by taking $x_1 = 0$ as the first estimate, then the third estimate (x_3) will be closest to

A 0.25079

B 0.25000

C 0.25039

D 0.25099

E 0.24784

Let $f(x) = x^4 - 4x + 1$

$f'(x) = 4x^3 - 4$

$$x_2 = 0 - \frac{f(0)}{f'(0)} = \frac{1}{4}$$

$$x_3 = \frac{1}{4} - \frac{f(\frac{1}{4})}{f'(\frac{1}{4})} = \frac{1}{4} - \frac{\left(\frac{1}{256}\right)}{\left(\frac{1}{16} - 4\right)}$$

$$= \frac{252}{1008} - \frac{1}{16 - 1024} = \frac{253}{1008}$$

8. *[4 marks]*

$$\lim_{x \to 1^+} \frac{x^{\frac{1}{2}} - 1}{e^x - ex} =$$

A $\frac{1}{2e}$

B $\frac{2}{e}$

C $2e$

D $\frac{e}{2}$

E ∞

As $x \to 1^+$, $x^{\frac{1}{2}} - 1 \to 0$ and $e^x - ex \to 0$ so the limit is indeterminate of type $\frac{0}{0}$ and equals $\lim_{x \to 1^+} \frac{\frac{1}{2}x^{-\frac{1}{2}}}{e^x - e}$ by L'Hôpital.

As $x \to 1^+$, $\frac{1}{2}x^{-\frac{1}{2}} \to \frac{1}{2}$ and $e^x - e \to 0^+$ so $\lim_{x \to 1^+} \frac{\frac{1}{2}x^{-\frac{1}{2}}}{e^x - e}$ is not indeterminate and is (positive) infinite.

9. *[4 marks]*

$\lim_{x \to \infty} (1 + \frac{1}{3}x^{-1})^{7+2x} =$ (call the limit y)

A 0

B 1

C e^7

D $e^{\frac{2}{3}}$

E 7

$$\ln y = \lim_{x\to\infty} \frac{\ln(1+\frac{1}{3}x^{-1})}{(7+2x)^{-1}} \quad (\text{type } \tfrac{0}{0})$$

$$= \lim_{x\to\infty} \frac{(1+\frac{1}{3}x^{-1})^{-1}\cdot(-\frac{1}{3}x^{-2})}{-(7+2x)^{-2}\cdot 2}$$

$$= \lim_{x\to\infty} (1+\tfrac{1}{3}x^{-1})^{-1}\frac{(7+2x)^2}{6x^2} = \lim_{x\to\infty}(1+\tfrac{1}{3}x^{-1})^{-1}\cdot\tfrac{1}{6}\cdot\left(\tfrac{7}{x}+2\right)^2$$

$= 1\cdot\frac{1}{6}\cdot 2^2$. With $\ln y = \frac{2}{3}$, $y = e^{\frac{2}{3}}$.

10. *[4 marks]*

The function $f(x) = \dfrac{x}{(x-1)^2}$ on the interval $[-2, 2]$ has

A an absolute minimum at $x = 2$ and an absolute maximum at $x = -2$

B an absolute minimum at $x = -2$ and no absolute maximum

C an absolute minimum at $x = -1$ and no absolute maximum

D an absolute minimum at $x = -1$ and an absolute maximum at $x = 2$

E no absolute minimum or maximum

$\lim_{x\to 1} f(x) = +\infty$ so f has no absolute maximum

$$f'(x) = \frac{(x-1)^2 - 2x(x-1)}{(x-1)^4} = \frac{1-x^2}{(x-1)^4}.$$

f decreases on $(-2,-1)$, increases on $(-1,1)$, and decreases on $(1,2)$ — and is continuous at -2 and 2. $f(-1) = -\frac{1}{4}$ and $f(2) = 2$; $-\frac{1}{4} < 2$.

PART B. Written-Answer Questions
SHOW YOUR WORK.

B1. *[15 marks]*

Solve the inequality $\dfrac{e^x - 2}{\ln x} > 0$.

Let $f(x) = \dfrac{e^x - 2}{\ln x}$ for $x > 0$ and $x \neq 1$

<u>Sign of $f(x)$:</u>

$e^x - 2 < 0$ $\quad$ $e^x - 2 > 0$ $\quad$ $e^x - 2 > 0$

$\ln x < 0$ $\quad$ $\ln x < 0$ $\quad$ $\ln x > 0$

$0 \quad \ln 2 \quad 1 \quad x$

$f(x) > 0$ $\quad$ $f(x) = 0$ $\quad$ $f(x) < 0$ $\quad$ $f(x) > 0$

Solutions of the inequality are

in $(0, \ln 2)$ and $(1, \infty)$:

$0 < x < \ln 2$ or $x > 1$

B2.(a) *[8 marks]*

If a demand function is given by $p = \dfrac{100 + 500q^{-\frac{1}{2}}}{175 + q^2}$, find the marginal revenue when $q = 25$.

Let $r(q) = q \cdot p(q)$ (revenue)

$$r(q) = \frac{100q + 500q^{\frac{1}{2}}}{175 + q^2}$$

$$r'(q) = \frac{(100 + 250q^{-\frac{1}{2}})(175 + q^2) - (100q + 500q^{\frac{1}{2}})(2q)}{(175 + q^2)^2}$$

$$r'(25) = \frac{150 \cdot 800 - 5000 \cdot 50}{(800)^2}$$

$$= \frac{12 - 25}{64}$$

$$= -\frac{13}{64}$$

B2.(b) *[7 marks]*

A refrigerator company finds that the average cost per refrigerator to produce q refrigerators is given in dollars by the function $\bar{c}(q) = 0.01q^2 - q + 70 + \dfrac{3000}{q}$.

(i) *[1 mark]* What is the total cost of producing q refrigerators?

(ii) *[3 marks]* What is the marginal cost if 10 refrigerators are made?

(iii) *[3 marks]* What is the relative rate of change of cost with respect to the number of refrigerators made when 10 refrigerators are made?

(i) Let $c(q)$ = total cost

$$c(q) = q\bar{c}(q) = 0.01q^3 - q^2 + 70q + 3000$$

(ii) $c'(q) = 0.03q^2 - 2q + 70$

$$c'(10) = 3 - 20 + 70 = 53$$

(iii) $c(10) = 10 - 100 + 700 + 3000$

$$= 3610$$

The relative rate of change of c when $q = 10$ is $\dfrac{c'(10)}{c(10)} = \dfrac{53}{3610}$

B3. *[15 marks]*

Let $f(x) = e^{\left(\frac{1}{x^2-1}\right)}$. Note that $f'(x) = \dfrac{-2x}{(x^2-1)^2}e^{\left(\frac{1}{x^2-1}\right)}$ and $f''^{(x)} = \dfrac{6x^4-2}{(x^2-1)^4}e^{\left(\frac{1}{x^2-1}\right)}$.

B3.(a) *[1 mark]*

Find $\lim\limits_{x\to 1^-} f(x)$.

As $x \to 1^-$, $x^2-1 \to 0^-$, $\frac{1}{x^2-1} \to -\infty$,

so that $\lim\limits_{x\to 1^-} e^{\frac{1}{x^2-1}} = 0$.

B3.(b) *[2 marks]*

Find all vertical and horizontal asymptotes of $y = f(x)$.

If either $x \to -1^-$ or $x \to 1^+$, then $x^2 \to 1^+$, $x^2-1 \to 0^+$, $\frac{1}{x^2-1} \to +\infty$, and $f(x) \to \infty$ so that $x=-1$ and $x=1$ are vertical asymptotes.

Also, if $x \to \pm\infty$, then $x^2-1 \to +\infty$, $\frac{1}{x^2-1} \to 0^+$, and $f(x) \to e^0 = 1$, so that $y=1$ is a horizontal asymptote.

B3.(c) *[3 marks]*

Find all intervals where f is increasing, intervals where f is decreasing, relative minima, and relative maxima.

($x=-1$ and $x=1$ are not in the domain of f; $f'(0)=0$)

Sign of f':

>0 -1 >0 0 <0 1 <0

f increases on $(-\infty,-1)$ and $(-1,0)$, decreases on $(0,1)$ and $(1,\infty)$.

Relative maximum at $x=0$.

(no relative minimum)

B3(d). *[3 marks]*

Find all intervals where f is concave up, intervals where f is concave down, and inflection points. Recall that $f(x) = e^{\left(\frac{1}{x^2-1}\right)}$, $f'(x) = \dfrac{-2x}{(x^2-1)^2}e^{\left(\frac{1}{x^2-1}\right)}$, and $f''(x) = \dfrac{6x^4-2}{(x^2-1)^4}e^{\left(\frac{1}{x^2-1}\right)}$.

Sign of f'':

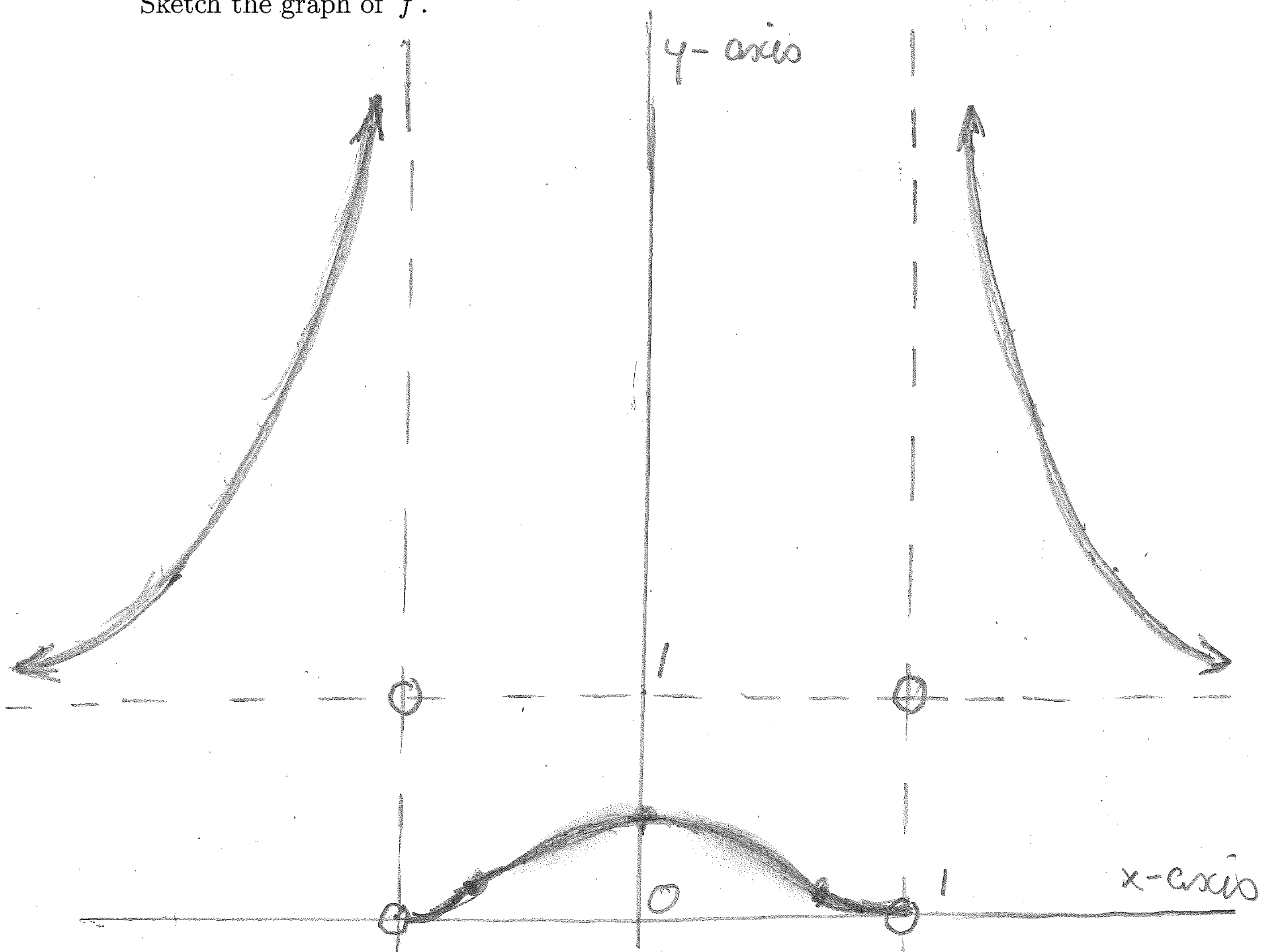
$>0 \quad >0 \quad <0 \quad >0 \quad >0$

$-1 \quad -3^{-\frac{1}{4}} \quad 3^{-\frac{1}{4}} \quad 1$

f is concave up on $(-\infty,-1)$, $(-1,-3^{-\frac{1}{4}})$, $(3^{-\frac{1}{4}},1)$, and $(1,\infty)$, concave down on $(-3^{-\frac{1}{4}}, 3^{-\frac{1}{4}})$. Inflection at $x=-3^{-\frac{1}{4}}$ and $x=3^{-\frac{1}{4}}$.

B3(e). *[6 marks]*

Sketch the graph of f.



B4. *[15 marks]*

A company finds that to sell q units of its product it must set its price to the customer at $72 - 5q^{\frac{1}{3}}$ dollars per unit, while it costs 32 dollars to produce each unit.

B4.(a) *[12 marks]*

Find the number of units the company should produce to maximize its profit.

revenue $(q) = q(72 - 5q^{\frac{1}{3}})$; cost $(q) = 32q$.

Let $f(q)$ denote profit;

$f(q) =$ revenue $(q) -$ cost $(q) = 40q - 5q^{\frac{4}{3}}$.

To maximize f, $f'(q) = 40 - \frac{20}{3} q^{\frac{1}{3}}$,

so q is critical iff $\frac{20}{3} q^{\frac{1}{3}} = 40$

iff $q = 216$. (This would maximize f, if there is a maximum possible profit.)

B4.(b) *[3 marks]*

Verify that your answer to B4.(a) actually maximizes profit. (Two solutions are given)

If $q < 216$, $\frac{20}{3} q^{\frac{1}{3}} < \frac{20}{3}(216)^{\frac{1}{3}} = 40$, $f'(q) > 0$, and f increases.

If $q > 216$, the inequalities reverse and f decreases.

By the first derivative test, $f(216)$ is the absolute max.

$f''(q) = -\frac{20}{9} q^{-\frac{2}{3}} < 0$ for all $q > 0$.

(which implies $f'(q) > 0$ for $q < 216$, $f'(q) < 0$ for $q > 216$)