MAT 133Y1Y TERM TEST #2

THURSDAY, JULY 2, 2009 7:30 - 9:30 PM

FAMILY NAME: ______________________

GIVEN NAMES: ______________________

STUDENT NUMBER: ______________________

TUTORIAL ROOM: ______________________

Aids Allowed: Calculator with empty memory, to be supplied by the student. Absolutely no graphing calculators allowed.

Instructions: This test has 10 multiple choice questions worth 4 marks each and 5 written answer questions worth a total of 60 marks. For each multiple choice question, you may do your rough work in the test booklet, but you must record your answer by circling one of the letters A, B, C, D or E which appear on the front page of the test. A multiple choice question left blank, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the written answer solutions, present your solutions in the spaces provided. Use the back of the question pages for your rough work.

ANSWERS FOR MULTIPLE CHOICE **Circle the correct answer**					
1.	**A**	**B**	**C**	**D**	**E**
2.	**A**	**B**	**C**	**D**	**E**
3.	**A**	**B**	**C**	**D**	**E**
4.	**A**	**B**	**C**	**D**	**E**
5.	**A**	**B**	**C**	**D**	**E**
6.	**A**	**B**	**C**	**D**	**E**
7.	**A**	**B**	**C**	**D**	**E**
8.	**A**	**B**	**C**	**D**	**E**
9.	**A**	**B**	**C**	**D**	**E**
10.	**A**	**B**	**C**	**D**	**E**

GRADER'S REPORT	
Multiple Choice	/ 40
Question 11	/ 13
Question 12	/ 14
Question 13	/ 20
Question 14	/ 13
TOTAL	/100

1. If $f(x) = \dfrac{(x^2+3x)^{(3/2)}}{x-2}$ then $f'(1) =$

$$f'(x) = \frac{3}{2}(x^2+3x)^{1/2}(2x+3)(x-2)^{-1} - (x-2)^{-2}(x^2+3x)^{3/2}$$

A -11

B 15

C $5\sqrt{5}$

D $2\sqrt{5} - 8$

E -23

$$f'(1) = \frac{3}{2}\sqrt{4}(5)(-1) - (1)(\sqrt{4})^3$$

$$= -15 - 8$$

$$= -23$$

2. $\lim\limits_{x \to +\infty} (1/x)^{(e^{-x})} = e^0 = 1$

$$\lim_{x\to+\infty} \ln\left(\frac{1}{x}\right)^{(e^{-x})} = \lim_{x\to+\infty} e^{-x}\ln\left(\frac{1}{x}\right)$$

A 0

B 1

C -1

D e

E does not exist

$$= \lim_{x\to+\infty} \frac{-\ln x \to -\infty}{e^x \to +\infty}$$

$$= \lim_{x\to+\infty} \frac{-\frac{1}{x} \to 0^-}{e^x \to +\infty}$$

$$= 0$$

3. The solution to the inequality $\dfrac{2^x - 1}{x^2 - x - 2} \le 0$ is:

$$\frac{2^x-1}{(x-2)(x+1)}$$

A $x < -1$ or $0 < x < 2$

B $-1 < x < 0$ or $x > 2$

C $x < -1$ or $x > 2$

D $x < -1$ or $0 \le x < 2$

E $-1 < x \le 0$ or $x > 2$

$2^x - 1 = 0 \Rightarrow 2^x = 1 \Rightarrow x = 0.$

$(x-2)(x+1) = 0 \Rightarrow x = 2, -1.$

-ve $\quad -1 \quad$ +ve $\quad 0 \quad$ -ve $\quad 2 \quad$ +ve

Solution: $x < -1$ or $0 \le x < 2$.

4. $\displaystyle\lim_{h\to 0} \frac{[\ln(2+h)]^3 - [\ln 2]^3}{h} =$

A 0

B $\dfrac{3(\ln 2)^2}{2}$

C 1

D 3/2

E does not exist

$f'(2) = \frac{3}{2}[\ln 2]^2.$ (limit definition)

where $f(x) = [\ln x]^3 \Rightarrow f'(x) = 3[\ln x]^2 \cdot \frac{1}{x}$

or $\displaystyle\lim_{h\to 0} \frac{[\ln(2+h)]^3 - [\ln 2]^3}{h}$ $\to 0$, $\to 0$

$= \displaystyle\lim_{h\to 0} \frac{3[\ln(2+h)]^2 \left(\frac{1}{2+h}\right) - 0}{1}$

$= \frac{3}{2}[\ln 2]^2.$ (L'hopital's rule)

5. An electronics company sells 5000 TV sets per year and it costs the company $10/year to store a TV set. If the cost of placing an order is $40 , then how many TV sets should be ordered each time in order to minimize the inventory costs?

$V = 5000 \quad A = 10 \quad B = 40.$

A 200

B 100

C 50

D 1000

E 500

$X = \sqrt{\frac{2BV}{A}} = \sqrt{\frac{2(40)(5000)}{10}} = \sqrt{40000}$

$= 200.$

→ most economic order size

6. Given the average cost function: $\bar{c} = 12 - 6q + q^2$, the marginal cost function is increasing

A when $q > 3$

B when $q > 2$

C when $0 < q < 2$

D always

E never

$c = 12q - 6q^2 + q^3$

$mc = \frac{dc}{dq} = 12 - 12q + 3q^2$

$\frac{dmc}{dq} = -12 + 6q > 0$ when MC is increasing

$\Rightarrow q > 2.$

7. If Newton's Method is used to approximate a solution of the equation: $x^3 - x + 1 = 0$ with initial estimate $x_1 = -2$, then $x_2 =$

$f(x) = x^3 - x + 1$

$f'(x) = 3x^2 - 1$

A 1/5

B −27/11

C −2/5

D −17/11

E 17/11

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= -2 - \frac{f(-2)}{f'(-2)}$$

$$= -2 - \frac{(-5)}{11}$$

$$= \frac{-22+5}{11} = \frac{-17}{11}$$

8. If $f(x) = \dfrac{x^2}{9 - x^2}$, then on the interval $[-2,3)$ f has:

A an absolute maximum at $x = 0$, but no absolute minimum

B an absolute minimum at $x = -2$, and an absolute maximum at $x = 0$

C an absolute minimum at $x = 0$, and an absolute maximum at $x = -2$

D an absolute minimum at $x = 0$, but no absolute maximum

E no absolute maximum or minimum

$$f'(x) = 2x(9-x^2)^{-1} - (9-x^2)^{-2}(-2x)x^2$$

$$= 2x(9-x^2)^{-2}[9-x^2+x^2]$$

$$= \frac{18x}{(9-x^2)^2}$$

$f'(x) = 0 \Rightarrow x = 0$

$f(-2) = \frac{4}{5}$

$f(0) = 0$

as $x \to 3^-$

$$\frac{x^2 \to 9}{9-x^2 \to 0^+} \to +\infty$$

9. If the demand for a product is: $p = -0.01q + 20$ and $q = 10\sqrt{m^2+3600} - 600$, (where p is the price when q units are produced by m employees), then the Marginal Revenue Product when $m = 80$ is:

A 147.2

B 128

C 8

D 12

E 96

$r = -0.01q^2 + 20q$

$$\frac{dr}{dm} = \frac{dr}{dq} \cdot \frac{dq}{dm}$$

$$= (-0.02q + 20)\left[5(m^2+3600)^{-1/2}(2m)\right]$$

$$m = 80 \Rightarrow q = 10\sqrt{10000} - 600 = 400.$$

$$MRP\Big|_{\substack{m=80 \\ q=400}} = (-8+20)\left[5 \frac{1}{100}(160)\right]$$

$$= 12[8]$$

$$= 96$$

10. If $C = 8 + \frac{I}{4} - \frac{\sqrt{I}}{2}$ where I is the national income and C is the consumption, then the Marginal Propensity to Save when $I = 9$ is:

A 5/6

B 3/4

C 1/12

D 1/6

E 1/4

$$MPC = \frac{dc}{dq} = \frac{1}{4} - \frac{1}{4} I^{-1/2}$$

$$MPC\Big|_{I=9} = \frac{1}{4} - \frac{1}{4\sqrt{9}} = \frac{3}{12} - \frac{1}{12} = \frac{1}{6}$$

$$MPS\Big|_{I=9} = 1 - MPC = 1 - \frac{1}{6} = \frac{5}{6}.$$

11.

$$\text{Given:} \quad f(x) = \begin{cases} 2^{\frac{1}{(1-x)}} + 1 & \text{if } x > 1 \\ x^2 + a & \text{if } x = 1 \\ \dfrac{b(\sqrt{x} - 1)}{x - 1} & \text{if } 0 < x < 1 \\ \dfrac{\sqrt{1+x^4}}{2x^2 - 1} & \text{if } x \le 0 \end{cases}$$

(6) (a) What values of a and b make f continuous at $x = 1$? Show all steps in your solution.

① $f(1) = 1 + a$ ①

② $\lim\limits_{x \to 1^+} \underbrace{2^{\overset{\to -\infty}{\frac{1}{1-x}}}}_{0} + 1 = 1$ ①

$\lim\limits_{x \to 1^-} \dfrac{b(\sqrt{x}-1)}{x-1} = \lim\limits_{x \to 1^-} \dfrac{b(x-1)}{(x-1)(\sqrt{x}+1)} = \lim\limits_{x \to 1^-} \dfrac{b}{\sqrt{x}+1} = \dfrac{b}{2}$ ②

If continuous at $x=1$

$1 + a = 1 = \dfrac{b}{2} \Rightarrow \begin{cases} a = 0 \\ b = 2 \end{cases}$ ②

(4) (b) Using the values of a and b found in (a), will f be continuous at $x = 0$? Justify your answer.

① $f(0) = \dfrac{1}{-1} = -1$ ①

② $\lim\limits_{x \to 0^+} \dfrac{2(\sqrt{x}-1)}{x-1} = 2$. $\lim\limits_{x \to 0^-} \dfrac{\sqrt{1+x^4}}{2x^2-1} = \dfrac{1}{-1} = -1$ ②

$\therefore \lim\limits_{x \to 0} f(x)$ does not exist $\Rightarrow$ f is not continuous at $x=0$. ①

(3) (c) What happens to the value of f(x) as $x \to -\infty$? Justify your answer.

when $x \to -\infty$ $f(x) = \dfrac{\sqrt{1+x^4}}{2x^2-1} \begin{matrix} \to +\infty \\ \to +\infty \end{matrix} = \dfrac{\sqrt{\frac{1}{x^4}+1}}{2-\frac{1}{x^2}} \to \dfrac{1}{2}$ ① ②

$\therefore$ f(x) approaches $\dfrac{1}{2}$.

12. (a) Given: $y^2 = (x^2+1)^x e^{-2x}$

Find the equation of the tangent to the above curve at the point $(0, -1)$.

$$\ln y^2 = \ln(x^2+1)^x + \ln e^{-2x} \quad 1$$

$$2\ln y = x\ln(x^2+1) - 2x \quad 1$$

$$\frac{2}{y}\frac{dy}{dx} = \ln(x^2+1) + \frac{1}{x^2+1}(2x)(x) - 2 \quad 2$$

(7) $(0,-1) \Rightarrow -2\frac{dy}{dx} = 0 + 0 - 2 \Rightarrow \frac{dy}{dx} = 1$ = Slope of tangent at $(0,-1)$ 2

equation of tangent at $(0,-1)$ is

$y+1 = 1(x-0)$ or $y = x-1$ 1

(b) Given: $x^2 e^{(-y^2)} = 1$ Show that $\frac{d^2y}{dx^2} = \frac{-(y^2+1)}{x^2y^3}$

$$2xe^{-y^2} + e^{-y^2}(-2y)\frac{dy}{dx}x^2 = 0 \quad 2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xe^{-y^2}}{2yx^2e^{-y^2}} = \frac{1}{xy} \text{ or } (xy)^{-1} \quad 2$$

(7) $$\frac{d^2y}{dx^2} = -(xy)^{-2}\left[y + \frac{dy}{dx}x\right] = \frac{-1}{(xy)^2}\left[y + \frac{x}{xy}\right] \quad 2$$

$$= \frac{-1}{x^2y^2}\left[y + \frac{1}{y}\right] = \frac{-(y^2+1)}{x^2y^3} \quad 1$$

13. Given: $f(x) = \frac{1+x^3}{1-x^3}$ $= (1+x^3)(1-x^3)^{-1}$

(a) Find all intercepts of f. $(0,1)$ $(-1,0)$ 2

(2)

(b) Find the horizontal and vertical asymptotes of f. Justify your answer in each case.

f is not defined at $x=1$

$x \to 1^+$ $\frac{1+x^3 \nearrow 2}{1-x^3 \searrow 0^-} \to -\infty$

$x \to 1^-$ $\frac{1+x^3 \to 2}{1-x^3 \searrow 0^+} \to +\infty$

$\therefore$ $x=1$ is a vertical asymptote 2

(4)

$x \to +\infty$ $\frac{1+x^3}{1-x^3} = \frac{\frac{1}{x^3}+1}{\frac{1}{x^3}-1} \to -1^-$ ($\frac{1}{x^3} \to 0^+$)

$x \to -\infty$ $\frac{1+x^3}{1-x^3} = \frac{\frac{1}{x^3}+1}{\frac{1}{x^3}-1} \to -1^+$ ($\frac{1}{x^3} \to 0^-$)

$\therefore$ $y=-1$ is a horizontal asymptote 2

(b) Show that $f'(x) = \frac{6x^2}{(1-x^3)^2}$ State where f is increasing and decreasing and find all relative maximum and minimum points.

$$f'(x) = 3x^2(1-x^3)^{-1} - (1-x^3)^{-2}(-3x^2)(1+x^3)$$
$$= 3x^2(1-x^3)^{-2}\left[(1-x^3)+(1+x^3)\right]$$
$$= \frac{6x^2}{(1-x^3)^2} > 0 \text{ for all } x \neq +1$$ 2

(4)

$\therefore f$ is always increasing 2

$\Rightarrow$ no relative extrema

13. (c) Show that $f''(x) = \dfrac{12x\,(1+2x^3)}{(1-x^3)^3}$ State where f is concave upward and concave downward and find all inflection points.

(6)

$$f''(x) = 12x(1-x^3)^{-2} - 2(1-x^3)^{-3}(-3x^2)(6x^2)$$
$$= 12x(1-x^3)^{-3}\left[(1-x^3)+3x^3\right]$$
$$= \frac{12x(1+2x^3)}{(1-x^3)^3} \quad ②$$

$f''(x)$ is not defined at $x=1$

$f''(x)=0 \Rightarrow x=0,\ -\sqrt[3]{\tfrac{1}{2}}$

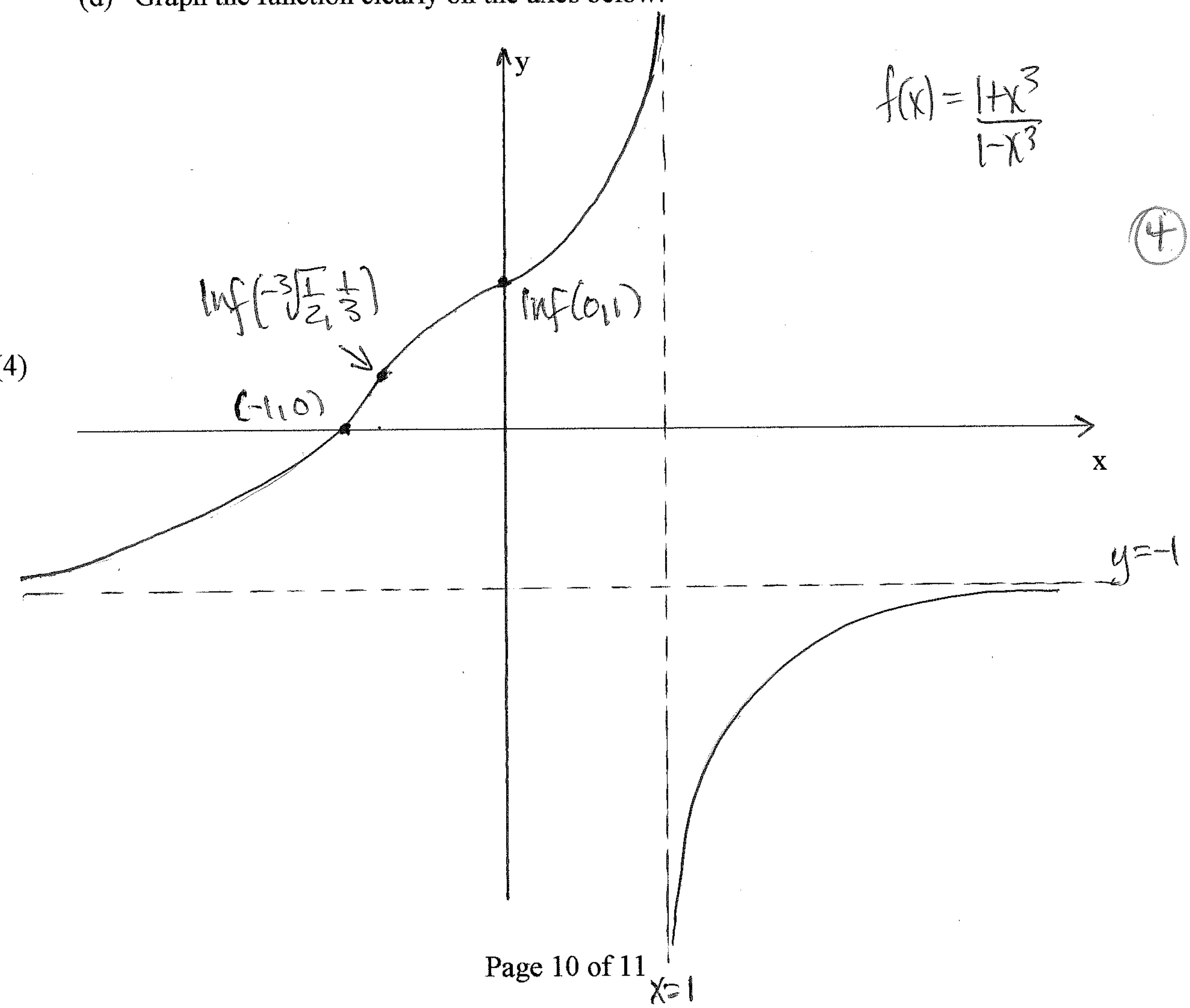
$f''(-1)>0$, $f''(-\tfrac{1}{2})<0$, $f''(\tfrac{1}{2})>0$, $f''(2)<0$

con up | $-\sqrt[3]{\tfrac{1}{2}}$ | con down | 0 | con up | 1 | con down ③

Inflection Pts $\left(-\sqrt[3]{\tfrac{1}{2}}, \tfrac{1}{3}\right)$ $(0,1)$ ①

(d) Graph the function clearly on the axes below:

(4)



14. Suppose the demand equation for a monopolist's product is: $p = 400 - 2q$ $\Rightarrow r = 400q - 2q^2$
and the average cost function is given by: $\bar{c} = q + 150 + \dfrac{200}{q}$, where q is the number of units produced and p & $\bar{c}$ are in dollars/unit. $\Rightarrow c = q^2 + 150q + 200$.

(a) If the government imposes a tax of $10/unit on the product, then what price should the monopolist charge in order to maximize his profits?

maximize : Profit = Revenue − Cost − tax cost. (1)

$$P = (400q - 2q^2) - (q^2 + 150q + 200 + 10q) \quad (3)$$

$$= -3q^2 + 240q - 200 \quad (1)$$

$$\frac{dP}{dq} = -6q + 240 = 0 \Rightarrow q = 40. \quad (2)$$

(10) $$\Rightarrow p = 320. \quad (1)$$

$$\frac{d^2P}{dq^2} = -6 < 0 \text{ for all values of } q. \quad (2)$$

$\Rightarrow$ absolute maximum when $q = 40$

$\therefore$ He should charge $320/unit in order to maximize his profits

(b) Find the point elasticity of demand for the product in (a) when q = 50. Is the demand elastic?

$$\eta = \frac{p/q}{dp/dq} = \frac{\frac{400-2q}{q}}{-2} = \frac{-200}{q} + 1 \quad (1)$$

(3) $$\eta\Big|_{q=50} = -4 + 1 = -3. \quad (1)$$

$\therefore$ demand is elastic (1)