MAT 133Y1Y TERM TEST 3

Thursday, 25 July, 2013, 7:10 pm – 9:10 pm

FAMILY NAME ______________________ Solutions

GIVEN NAME(S) ______________________

STUDENT NO. ______________________

SIGNATURE ______________________

GRADER'S REPORT	
Question	Mark
MC/40	
B1/15	
B2/15	
B3/15	
B4/15	
TOTAL	

NOTE:

1. **Aids Allowed:** Calculator with empty memory, to be supplied by student. Absolutely no graphing calculators allowed.
2. **Instructions:** Fill in the information on this page and ensure that the test contains 10 pages.
3. This test has 10 multiple choice questions worth 4 marks each and 4 written-answer questions worth 15 marks each.

   For the **multiple choice questions** indicate your answers by circling the appropriate letters (**A**, **B**, **C**, **D**, or **E**) on **this page (page 1)**. A multiple choice question left blank on **this page**, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the **written-answer questions**, present your solutions in the spaces provided. Use the multiple choice question pages or the back of any of the pages for rough work, for any of the questions.

ANSWER BOX FOR PART A

Circle the correct answer.

1.	A	B	C	D	E
2.	A	B	C	D	E
3.	A	B	C	D	E
4.	A	B	C	D	E
5.	A	B	C	D	E
6.	A	B	C	D	E
7.	A	B	C	D	E
8.	A	B	C	D	E
9.	A	B	C	D	E
10.	A	B	C	D	E

PART A. Multiple Choice

1. *[4 marks]*

   If differential approximation is used to estimate a value for $(4+h)^{\frac{3}{2}}$ when h is near 0, the result is

   A $8+4h$

   B $8+\frac{3}{2}h$

   C $8+h^{\frac{3}{2}}$

   D $8+\frac{3}{2}h^{\frac{1}{2}}$

   E $8+3h$

Let $f(x) = x^{\frac{3}{2}}$

(so that $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$, $f(4)=8$, and $f'(4)=3$)

If $(4+h, y)$ lies on the line which is tangent, at $(4,8)$, to the graph of f, then $y = f(4) + f'(4)\cdot h$ and this is the differential approximation to $f(4+h) = (4+h)^{\frac{3}{2}}$.

2. *[4 marks]*

   If $f''(x) = e^x - 1$, $f(0)=1$, and $f'(0)=-1$, then $f(x) =$

   A $e^x - \frac{1}{2}x^2 - 2x$

   B $e^x - \frac{1}{2}x^2 - x + 1$

   C $e^x - x^2 - 2x$

   D $e^x - \frac{1}{2}x^2$

   E $e^x - x^2 + x$

$f'(x) = \int f''(x)\,dx$

$= \int e^x - 1\,dx = e^x - x + C_1$

where C_1 satisfies $-1 = e^0 - 0 + C_1$

so that $C_1 = -2$ and $f'(x) = e^x - x - 2$.

$f(x) = \int f'(x)\,dx = \int e^x - x - 2\,dx = e^x - \frac{1}{2}x^2 - 2x + C_0$

where C_0 satisfies $1 = e^0 - 0 - 0 + C_0$

so that $C_0 = 0$.

3. *[4 marks]*

If $f(x) = \int_e^x \frac{dt}{\ln t}$, then $f'(e^3) =$

A $\frac{1}{3}$

B $-\frac{1}{3}e^{-3}$

C $-\frac{1}{9}e^{-3}$

D $3e^2$

E 1

Fundamental theorem:

$$\frac{1}{\ln(e^3)} = \frac{1}{3}$$

4. *[4 marks]*

If $F'(x) = f(x)$, $F(0) = -2$, and $F(2) = 3$, then $\int_0^2 f(x)\, dx =$

A -5

B 5

C 2

D -2

E 1

Fundamental theorem:

$$F(2) - F(0) = 3 - (-2)$$

5. *[4 marks]*

If $f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 2 & \text{if } 1 < x \leq 2 \\ 4 - x & \text{if } 2 < x \leq 4 \end{cases}$, then $\int_0^4 f(x)\,dx =$

A 7

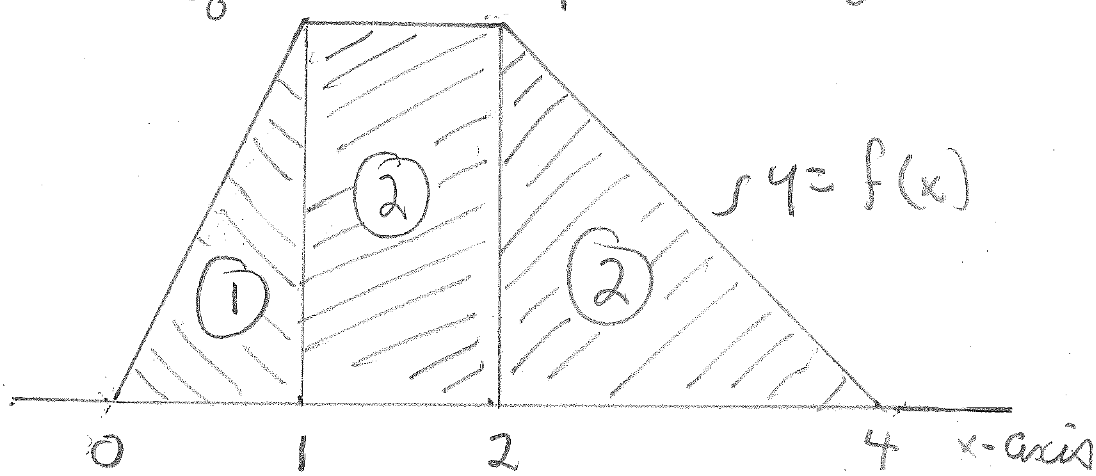
B 4

C $\frac{11}{2}$

D 6

E 5

$$\int_0^1 2x\,dx + \int_1^2 2\,dx + \int_2^4 4-x\,dx$$



6. *[4 marks]*

If a certain good has demand function $p = 15 - q^2$ and supply function $p = 2q$, what is the consumers' surplus for the good? Note: this good has equilibrium point $q = 3$, $p = 6$.

A 27

B 24

C 18

D 12

E 30

$$\int_0^3 (15 - q^2) - 6\,dq$$

$$= \left[9q - \frac{q^3}{3}\right]_0^3$$

$$= 9 \cdot 3 - \frac{3^3}{3}$$

7. *[4 marks]*

The profit from a business at time t is estimated to accrue at a continuous rate of $\$20,000e^{.005t}$ per year. If interest is 6% compounded continuously over the next 5 years, then the present value of all profit for the next 5 years (to the nearest $10) is

A $87,430

B $87,490

C $87,470

D $87,450

E $87,510

$$\int_0^5 20{,}000e^{.005t}e^{-.06t}\,dt$$

$$= \int_0^5 20{,}000\,e^{-.055t}\,dt$$

$$= \left[\frac{20{,}000}{-.055}e^{-.055t}\right]_0^5$$

$$= \frac{20{,}000}{.055}\left(1-e^{-.055\cdot 5}\right) = 87{,}428.32$$

8. *[4 marks]*

The average value, on $[-4, 2]$, of $f(x) = x^3$ is

A -9

B -10

C -16

D $-\frac{28}{3}$

E -12

$$\frac{1}{2-(-4)}\int_{-4}^{2} x^3\,dx = \frac{1}{6}\left[\frac{x^4}{4}\right]_{-4}^{2}$$

$$= \frac{1}{24}\left(2^4-(-4)^4\right)$$

9. *[4 marks]*

$$\int_2^\infty \frac{x-1}{x^3}\,dx \quad = \lim_{b\to\infty}\left[\int_2^b x^{-2} - x^{-3}\,dx\right]$$

A $= \frac{1}{4}$

B diverges

C $= \frac{3}{8}$

D $= \frac{7}{16}$

E $= \frac{1}{2}$

$$= \lim_{b\to\infty}\left[-x^{-1} + \frac{1}{2}x^{-2}\right]_2^b$$

$$= \lim_{b\to\infty} -b^{-1} + \frac{1}{2}b^{-2} + 2^{-1} - \frac{1}{2}\cdot 2^{-2}\Big]$$

$= \frac{1}{2} - \frac{1}{8}$ because as $b \to \infty$, both $b^{-1} \to 0$ and $b^{-2} \to \infty$

10. *[4 marks]*

If $\dfrac{dy}{dx} = (\ln 5)\cdot y$ and $y(0) = 2$, then $y(3) =$

A 225

B 216

C 288

D 250

E 240

Separating variables, $\frac{dy}{y} = \ln 5\, dx$. Integrating both sides, $\ln|y| = C + (\ln 5)\cdot x$.

Since $|y| = 2$ when $x = 0$, $C = \ln 2$ and $\ln|y| = \ln 2 + (\ln 5)\cdot x$, so that $|y| = 2\cdot 5^x$.

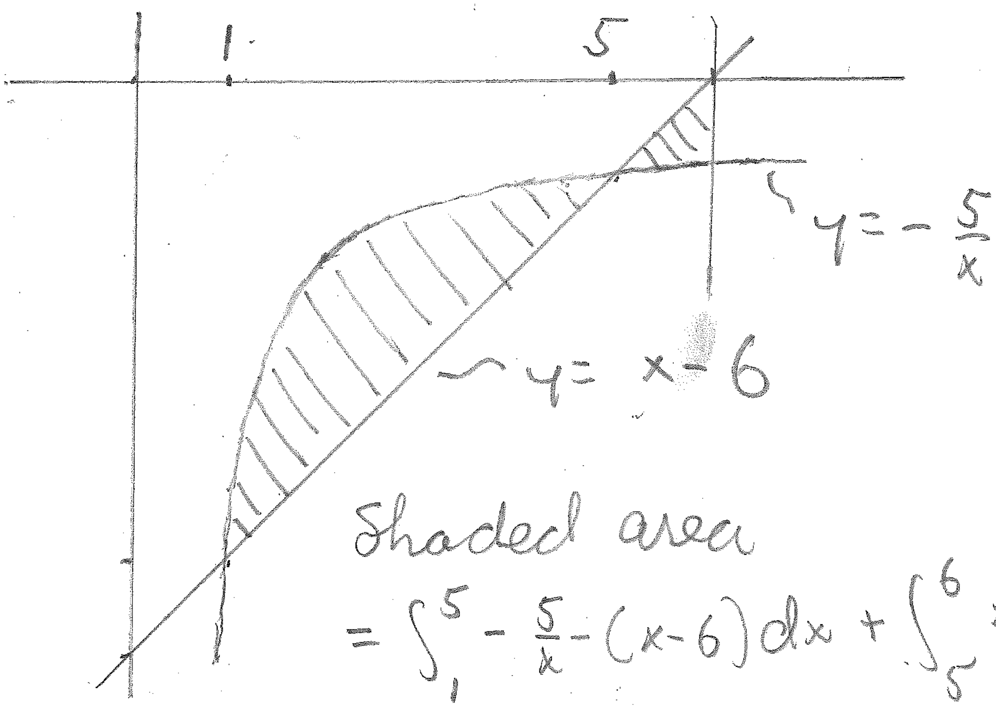
When $x = 3$, $|y| = 2\cdot 5^3$.

PART B. Written-Answer Questions
SHOW YOUR WORK.

B1. *[15 marks]*

Find the total area of the region(s) bounded by the curve $xy = -5$ and the lines $x = 6$ and $y = x - 6$.

$xy = -5$ iff $y = -\frac{5}{x}$. The graphs of $y = -\frac{5}{x}$ and $y = x - 6$ intersect provided $x - 6 = -\frac{5}{x}$ and $x^2 - 6x + 5 = (x-1)(x-5) = 0$.



Shaded area

$$= \int_1^5 -\frac{5}{x} - (x-6)\,dx + \int_5^6 x - 6 - \left(-\frac{5}{x}\right)dx$$

$$= \left[-5\ln x - \frac{x^2}{2} + 6x\right]_1^5 + \left[\frac{x^2}{2} - 6x + 5\ln x\right]_5^6$$

$$= \left[\left(-5\ln 5 - \frac{25}{2} + 30\right) - \left(-\frac{1}{2} + 6\right)\right] + \left[(18 - 36 + 5\ln 6) - \left(\frac{25}{2} - 30 + 5\ln 5\right)\right]$$

$$= \frac{23}{2} - 10\ln 5 + 5\ln 6$$

B2. *[15 marks]*

Find $\int x^2 e^{-\frac{1}{2}x}\,dx$.

By parts:

Let $u = x^2$, $dv = e^{-\frac{1}{2}x}\,dx$ so that $du = 2x\,dx$, $v = -2e^{-\frac{1}{2}x}$, and

$$\int x^2 e^{-\frac{1}{2}x}\,dx = -2x^2 e^{-\frac{1}{2}x} + 4\int x e^{-\frac{1}{2}x}\,dx$$

?

By parts again, with x and dx instead of u and du, and with dv and v as before,

$$\int x\, e^{-\frac{1}{2}x}\,dx = -2x e^{-\frac{1}{2}x} + 2\int e^{-\frac{1}{2}x}\,dx$$

$$= -2x e^{-\frac{1}{2}x} - 4e^{-\frac{1}{2}x} + \text{a constant}$$

Substituting this expression where indicated above, $\int x^2 e^{-\frac{1}{2}x}\,dx =$

$$-2x^2 e^{-\frac{1}{2}x} - 8x e^{-\frac{1}{2}x} - 16 e^{-\frac{1}{2}x} + \text{a constant} = -2(x^2+4x+8)e^{-\frac{1}{2}x} + \text{a constant}$$

B3. *[15 marks]*

Find $\displaystyle\int \frac{9}{(x+1)(x-2)^2}\,dx$.

If the partial fraction decomposition of the integrand is $\dfrac{A}{x+1} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$, then A, B, and C satisfy

$A(x-2)^2 + B(x+1)(x-2) + C(x+1) = 9$

for all x. In particular, with $x=-1$, $A(-1-2)^2 = 9$ so $A=1$. With $x=2$, $C(2+1) = 9$ so $C=3$. With $x=0$, $A(-2)^2 + B(-2) + C = 9$, $4A + C - 9 = 2B$, $4 + 3 - 9 = 2B$, and $B=-1$.

The required integral is

$$\int \frac{1}{x+1} - \frac{1}{x-2} + \frac{3}{(x-2)^2}\,dx =$$

$$\ln|x+1| - \ln|x-2| - \frac{3}{x-2} + \text{a constant} = \ln\left|\frac{x+1}{x-2}\right| - \frac{3}{x-2} + \text{a constant}$$

B4. *[15 marks]*

Find $y(x)$, defined for all $x > 1$ and expressed **explicitly** in terms of x, such that $(x-1)yy' = 1$ and $y(2) = 1$.

In differential form with variables separated, the equation is $y\,dy = \frac{dx}{x-1}$. Integrating both sides yields the general solution

$\frac{y^2}{2} = \ln(x-1) + C$, noting that $x>1$

and where the integration constant C is determined by the initial condition ($y=1$ when $x=2$): $\frac{1^2}{2} = \ln(2-1) + C$.

Thus $C = \frac{1}{2}$ and $y(x)$ is given implicitly by $\frac{y^2}{2} = \ln(x-1) + \frac{1}{2}$ or

$y^2 = 2\ln(x-1) + 1$. $y(x)$ is given explicitly by

$$y(x) = \sqrt{2\ln(x-1)+1} = \sqrt{\ln(x-1)^2+1}$$