# MAT 133Y1Y TERM TEST 3

Thursday, 23 July, 2015, 6:10 pm – 8:00 pm

Code 3

FAMILY NAME ______________________ Solutions

GIVEN NAME(S) ______________________

STUDENT NO. ______________________

SIGNATURE ______________________

| GRADER'S REPORT | |
|---|---|
| Question | Mark |
| MC/40 | |
| B1/15 | |
| B2/15 | |
| B3/15 | |
| B4/15 | |
| TOTAL | |

**NOTE:**

1. **Aids Allowed:** Calculator with empty memory, to be supplied by student. Absolutely no graphing calculators allowed.
2. **Instructions:** Fill in the information on this page and ensure that the test contains 10 pages.
3. This test has 10 multiple choice questions worth 4 marks each and 4 written-answer questions worth 15 marks each.

   For the **multiple choice questions** indicate your answers by circling the appropriate letters (**A**, **B**, **C**, **D**, or **E**) on **this page (page 1)**. A multiple choice question left blank on **this page**, or having an incorrect answer circled, or having more than one answer circled, will be assigned a mark of 0. For the **written-answer questions**, present your solutions in the spaces provided. Use the multiple choice question pages or the back of any of the pages for rough work, for any of the questions.

**ANSWER BOX FOR PART A**

**Circle the correct answer.**

| | | | | | |
|---|---|---|---|---|---|
| 1. | A | B | C | D | E |
| 2. | A | B | C | D | E |
| 3. | A | B | C | D | E |
| 4. | A | B | C | D | E |
| 5. | A | B | C | D | E |
| 6. | A | B | C | D | E |
| 7. | A | B | C | D | E |
| 8. | A | B | C | D | E |
| 9. | A | B | C | D | E |
| 10. | A | B | C | D | E |

## PART A. Multiple Choice

1. *[4 marks]*

If differential approximation is used to estimate a value for $(4+h)^{\frac{3}{2}}$ when $h$ is near $0$, the result is

**A** $8+4h$

**B** $8+\frac{3}{2}h$

**C** $8+h^{\frac{3}{2}}$

**D** $8+\frac{3}{2}h^{\frac{1}{2}}$

**E** $8+3h$

Let $f(x) = x^{\frac{3}{2}}$

(so that $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$, $f(4)=8$, and $f'(4)=3$)

If $(4+h, y)$ lies on the line which is tangent, at $(4,8)$, to the graph of $f$, then $y = f(4) + f'(4)\cdot h$ and this is the differential approximation to $f(4+h) = (4+h)^{\frac{3}{2}}$.

2. *[4 marks]*

If $f''(x) = e^x - 1$, $f(0) = 1$, and $f'(0) = -1$, then $f(x) =$

**A** $e^x - \frac{1}{2}x^2 - 2x$

**B** $e^x - \frac{1}{2}x^2 - x + 1$

**C** $e^x - x^2 - 2x$

**D** $e^x - \frac{1}{2}x^2$

**E** $e^x - x^2 + x$

$f'(x) = \int f''(x)\,dx$

$= \int e^x - 1\,dx = e^x - x + C_1$

where $C_1$ satisfies $-1 = e^0 - 0 + C_1$

so that $C_1 = -2$ and $f'(x) = e^x - x - 2$.

$f(x) = \int f'(x)\,dx = \int e^x - x - 2\,dx = e^x - \frac{1}{2}x^2 - 2x + C_0$

where $C_0$ satisfies $1 = e^0 - 0 - 0 + C_0$

so that $C_0 = 0$.

3. *[4 marks]*

If $f(x) = \int_e^x \frac{dt}{\ln t}$, then $f'(e^3) =$

A $\frac{1}{3}$

B $-\frac{1}{3}e^{-3}$

C $-\frac{1}{9}e^{-3}$

D $3e^2$

E $1$

Fundamental theorem:

$$\frac{1}{\ln(e^3)} = \frac{1}{3}$$

4. *[4 marks]*

If $F'(x) = f(x)$, $F(0) = -2$, and $F(2) = 3$, then $\int_0^2 f(x)\,dx =$

A $-5$

B $5$

C $2$

D $-2$

E $1$

Fundamental theorem:

$$F(2) - F(0) = 3 - (-2)$$

5. *[4 marks]*

If $f(x) = \begin{cases} 2x & \text{if} \quad 0 \le x \le 1 \\ 2 & \text{if} \quad 1 < x \le 2 \\ 4 - x & \text{if} \quad 2 < x \le 4 \end{cases}$ , then $\int_0^4 f(x)\,dx =$

**A** 7

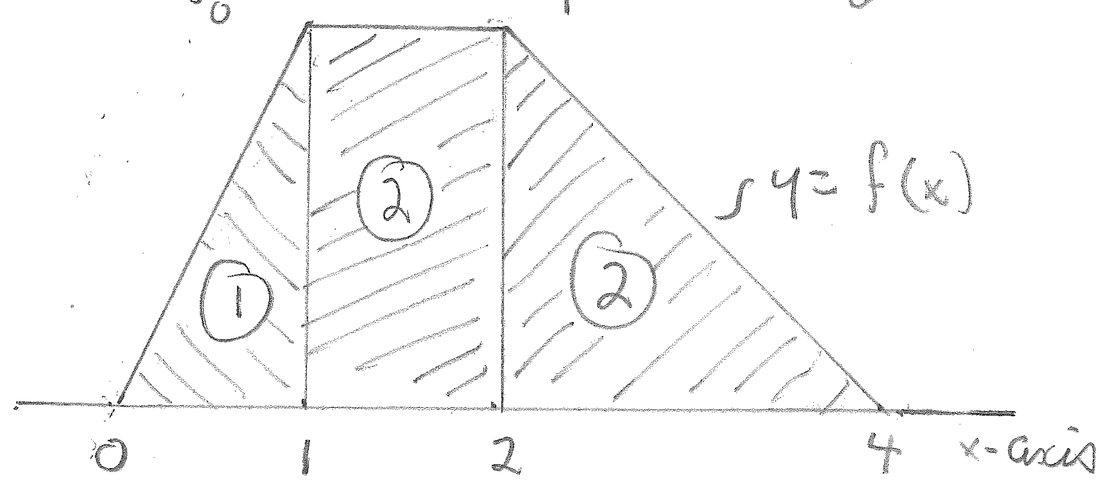
**B** 4

**C** $\frac{11}{2}$

**D** 6

**E** 5

$$\int_0^1 2x\,dx + \int_1^2 2\,dx + \int_2^4 4-x\,dx$$




6. *[4 marks]*

If a certain good has demand function $p = 15 - q^2$ and supply function $p = 2q$ , what is the consumers' surplus for the good? Note: this good has equilibrium point $q = 3$ , $p = 6$ .

**A** 27

**B** 24

**C** 18

**D** 12

**E** 30

$$\int_0^3 (15 - q^2) - 6\,dq$$

$$= \left[ 9q - \frac{q^3}{3} \right]_0^3$$

$$= 9 \cdot 3 - \frac{3^3}{3}$$

7. *[4 marks]*

The profit from a business at time $t$ is estimated to accrue at a continuous rate of $\$20,000e^{.005t}$ per year. If interest is 6% compounded continuously over the next 5 years, then the present value of all profit for the next 5 years (to the nearest \$10) is

A \$87,430

B \$87,490

C \$87,470

D \$87,450

E \$87,510

$$\int_0^5 20{,}000 e^{.005t} e^{-.06t}\,dt$$

$$= \int_0^5 20{,}000\, e^{-.055t}\,dt$$

$$= \left[\frac{20000}{-.055} e^{-.055t}\right]_0^5$$

$$= \frac{20{,}000}{.055}\left(1 - e^{-.055\cdot 5}\right) = 87{,}428.32$$

8. *[4 marks]*

The average value, on $[-4, 2]$, of $f(x) = x^3$ is

A $-9$

B $-10$

C $-16$

D $-\frac{28}{3}$

E $-12$

$$\frac{1}{2-(-4)} \int_{-4}^{2} x^3\,dx = \frac{1}{6}\left[\frac{x^4}{4}\right]_{-4}^{2}$$

$$= \frac{1}{24}\left(2^4 - (-4)^4\right)$$

9. *[4 marks]*
$$\int_2^{\infty} \frac{x-1}{x^3}\,dx$$

**A** $= \frac{1}{4}$

**B** diverges

**C** $= \frac{3}{8}$

**D** $= \frac{7}{16}$

**E** $= \frac{1}{2}$

$$= \lim_{b\to\infty}\left[\int_2^b x^{-2} - x^{-3}\,dx\right]$$

$$= \lim_{b\to\infty}\left[-x^{-1} + \frac{1}{2}x^{-2}\right]_2^b$$

$$= \lim_{b\to\infty} -b^{-1} + \frac{1}{2}b^{-2} + 2^{-1} - \frac{1}{2}\cdot 2^{-2}\Big]$$

$= \frac{1}{2} - \frac{1}{8}$ because as $b \to \infty$, both $b^{-1} \to 0$ and $b^{-2} \to \infty$

10. *[4 marks]*
If $\dfrac{dy}{dx} = (\ln 5)\cdot y$ and $y(0) = 2$, then $y(3) =$

**A** 225

**B** 216

**C** 288

**D** 250

**E** 240

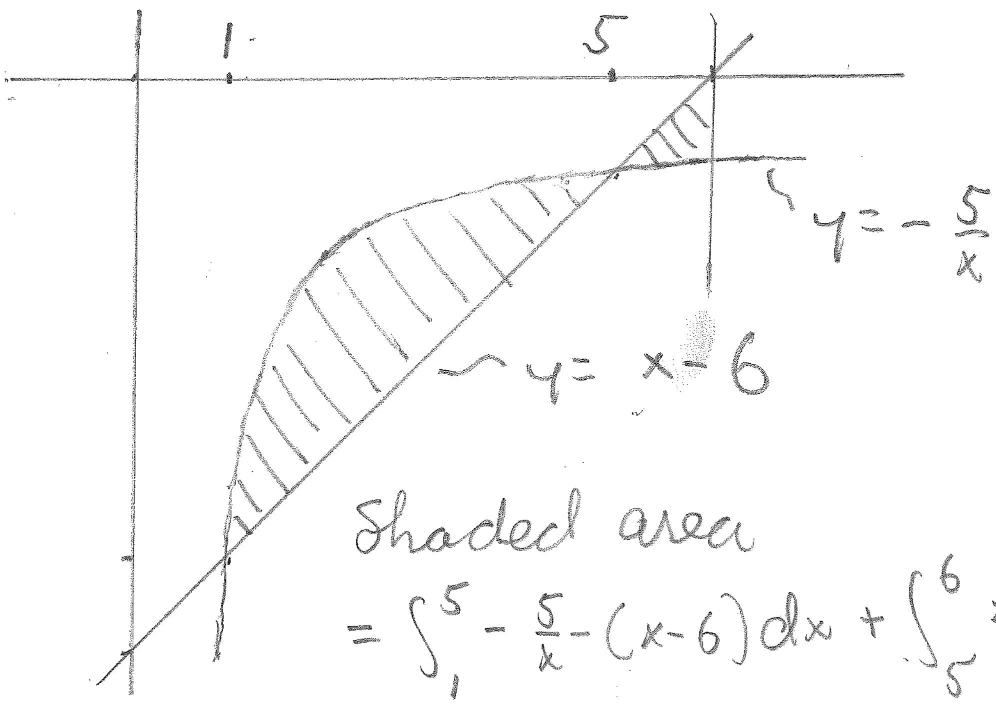
Separating variables, $\frac{dy}{y} = \ln 5\, dx$. Integrating both sides, $\ln|y| = C + (\ln 5)\cdot x$.
Since $|y| = 2$ when $x = 0$, $C = \ln 2$ and $\ln|y| = \ln 2 + (\ln 5)\cdot x$, so that $|y| = 2\cdot 5^x$.
When $x = 3$, $|y| = 2\cdot 5^3$.

## PART B. Written-Answer Questions
## SHOW YOUR WORK.

B1. *[15 marks]*

Find the total area of the region(s) bounded by the curve $xy = -5$ and the lines $x = 6$ and $y = x - 6$.

$xy = -5$ iff $y = -\frac{5}{x}$. The graphs of $y = -\frac{5}{x}$ and $y = x - 6$ intersect provided $x - 6 = -\frac{5}{x}$ and $x^2 - 6x + 5 = (x-1)(x-5) = 0$.




Shaded area

$$= \int_1^5 -\frac{5}{x} - (x-6)\,dx + \int_5^6 x - 6 - \left(-\frac{5}{x}\right) dx$$

$$= \left[-5\ln x - \frac{x^2}{2} + 6x\right]_1^5 + \left[\frac{x^2}{2} - 6x + 5\ln x\right]_5^6$$

$$= \left[\left(-5\ln 5 - \frac{25}{2} + 30\right) - \left(-\frac{1}{2} + 6\right)\right] + \left[(18 - 36 + 5\ln 6) - \left(\frac{25}{2} - 30 + 5\ln 5\right)\right]$$

$$= \frac{23}{2} - 10\ln 5 + 5\ln 6$$

B2. *[15 marks]*

Find $\int x^2 e^{-\frac{1}{2}x}\,dx$.

By parts:

Let $u = x^2$, $dv = e^{-\frac{1}{2}x}dx$ so that $du = 2x\,dx$, $v = -2e^{-\frac{1}{2}x}$, and

$$\int x^2 e^{-\frac{1}{2}x}dx = -2x^2 e^{-\frac{1}{2}x} + 4\int x e^{-\frac{1}{2}x}dx \quad ?$$

By parts again, with $x$ and $dx$ instead of $u$ and $du$, and with $dv$ and $v$ as before,

$$\int x\, e^{-\frac{1}{2}x}dx = -2x e^{-\frac{1}{2}x} + 2\int e^{-\frac{1}{2}x}dx$$

$$= -2x e^{-\frac{1}{2}x} - 4e^{-\frac{1}{2}x} + \text{a constant}$$

Substituting this expression where indicated above, $\int x^2 e^{-\frac{1}{2}x}dx =$

$$-2x^2 e^{-\frac{1}{2}x} - 8xe^{-\frac{1}{2}x} - 16e^{-\frac{1}{2}x} + \text{a constant} = -2(x^2+4x+8)e^{-\frac{1}{2}x} + \text{a constant}$$

B3. *[15 marks]*

Find $\displaystyle\int \frac{9}{(x+1)(x-2)^2}\,dx$.

If the partial fraction decomposition of the integrand is $\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$, then $A$, $B$, and $C$ satisfy

$A(x-2)^2 + B(x+1)(x-2) + C(x+1) = 9$

for all $x$. In particular, with $x=-1$, $A(-1-2)^2 = 9$ so $A=1$. With $x=2$, $C(2+1) = 9$ so $C=3$. With $x=0$, $A(-2)^2 + B(-2) + C = 9$, $4A + C - 9 = 2B$, $4+3-9 = 2B$, and $B=-1$.

The required integral is

$$\int \frac{1}{x+1} - \frac{1}{x-2} + \frac{3}{(x-2)^2}\,dx =$$

$$\boxed{\ln|x+1| - \ln|x-2| - \frac{3}{x-2} + \text{a constant}} = \boxed{\ln\left|\frac{x+1}{x-2}\right| - \frac{3}{x-2} + \text{a constant}}$$

B4. *[15 marks]*

Find $y(x)$, defined for all $x > 1$ and expressed **explicitly** in terms of $x$, such that $(x-1)yy' = 1$ and $y(2) = 1$.

In differential form with variables separated, the equation is $y\,dy = \frac{dx}{x-1}$. Integrating both sides yields the general solution $\frac{y^2}{2} = \ln(x-1) + C$, noting that $x > 1$ and where the integration constant $C$ is determined by the initial condition ($y = 1$ when $x = 2$): $\frac{1^2}{2} = \ln(2-1) + C$.

Thus $C = \frac{1}{2}$ and $y(x)$ is given implicitly by $\frac{y^2}{2} = \ln(x-1) + \frac{1}{2}$ or $y^2 = 2\ln(x-1) + 1$. $y(x)$ is given explicitly by

$$y(x) = \sqrt{2\ln(x-1)+1} = \sqrt{\ln(x-1)^2 + 1}$$