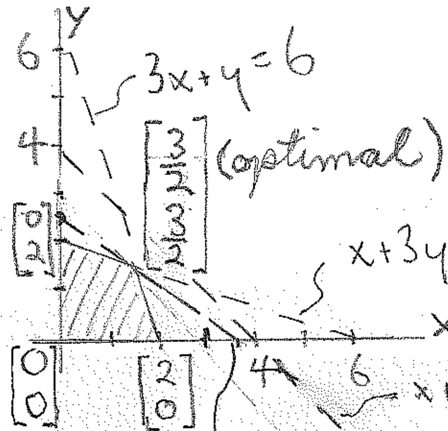
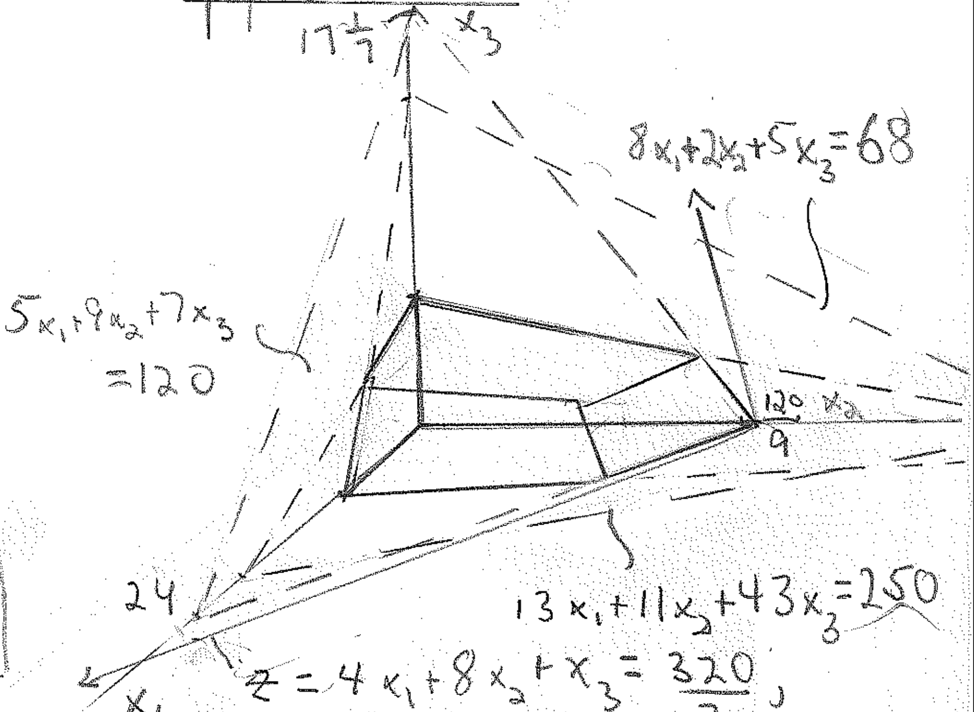
pg. 82 14.



$Z = 2x+3y = \frac{15}{2}$, optimal objective value

pg. 83 16.

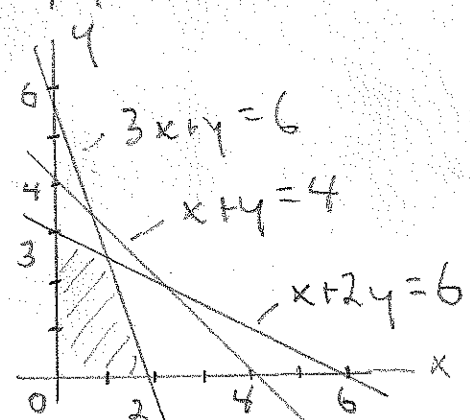


$z = 4x_1+8x_2+x_3 = \frac{320}{3}$, optimal objective value

optimal

Extreme points: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{17}{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{40}{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{250}{43} \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 11 \\ 3 \end{bmatrix}$

pg. 91 4.



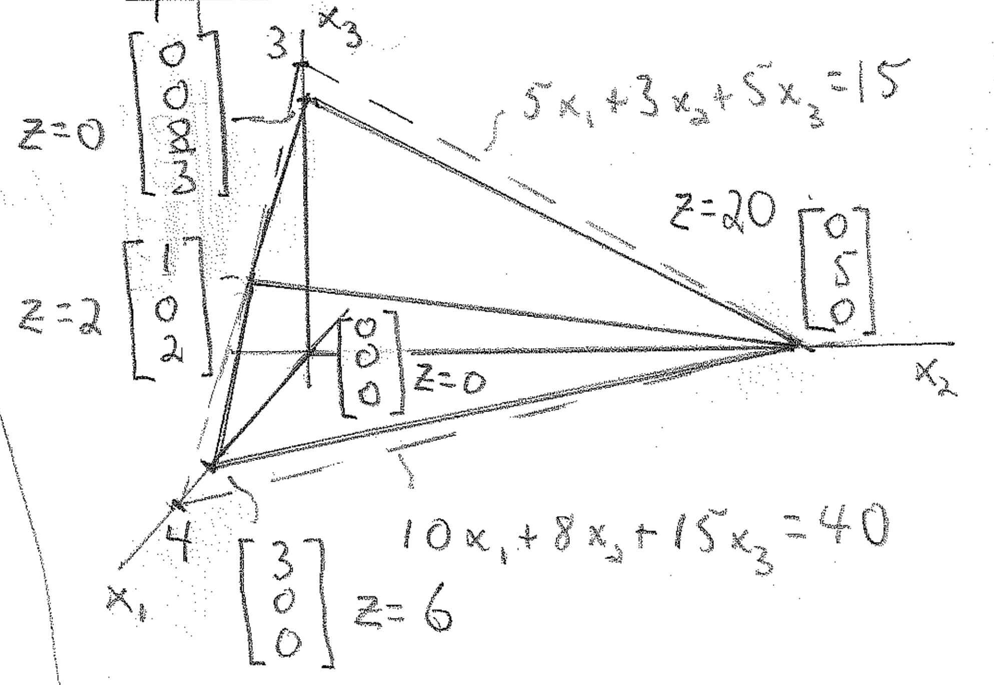
Extreme points

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, z=0$; $\begin{bmatrix} 2 \\ 0 \end{bmatrix}, z=4$;

$\begin{bmatrix} \frac{6}{5} \\ \frac{12}{5} \end{bmatrix}, z=\frac{48}{5}$; $\begin{bmatrix} 0 \\ 3 \end{bmatrix}, z=9$

optimal

pg. 91 8.



$\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$ is optimal.

<u>pg. 91 12.</u> This problem has an empty feasible region, which therefore has no extreme points. For, the first constraint implies $2x_1+2x_2+2x_3 \le 2$ for any feasible $[x_1 \ x_2 \ x_3]^T$. But $2x_1+2x_2+2x_3 \ge 2x_1+x_2+2x_3$ by the constraint $x_2 \ge 0$, and $2x_1+x_2+2x_3 \ge 3$ by the second constraint. Thus, the constraints imply that every feasible $[x_1 \ x_2 \ x_3]^T$ satisfy the (contradictory) conditions
$2x_1+2x_2+2x_3 \le 2$ and $2x_1+2x_2+2x_3 \ge 3$.

<u>pg. 100 6.</u> In each part (a) - (d), the transpose of the given 5-tuple will be referred to as "x".

(a) This x is not even a solution of the system $Ax=b$.

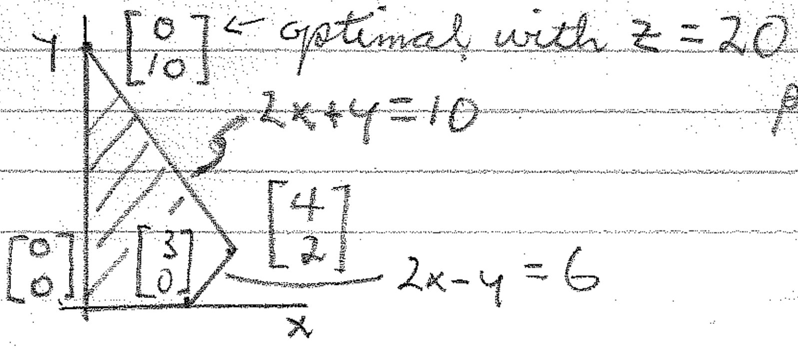
(b) This x is a solution of the system $Ax=b$ but is not a basic solution: $\left\{\begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}\right\}$ is linearly dependent.

(c) Also a solution, but not a basic solution. $\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}, \begin{bmatrix}4\\1\end{bmatrix}\right\}$ is linearly dependent.

(d) This x is a basic solution for the system $Ax=b$. The basic variables are x_3 and any one of x_1, x_4, or x_5.

<u>pg. 100 8.</u> In order to eventually solve (b), it is convenient to solve (c) first.

<u>(c)</u>



<u>(a)</u> Introducing slacks s_1, s_2, an equivalent canonical problem is: Maximize $z=3x+2y$

$$\text{s.t. } \begin{aligned} 2x-y+s_1 \quad &= 6 \\ 2x+y \quad +s_2 &= 10 \end{aligned}$$

$x \ge 0, y \ge 0, s_1 \ge 0, s_2 \ge 0$.

pg. 100 8. (b) Basic feasible solutions of the problem given as the solution of part (a) are:

$$\begin{bmatrix} x \\ y \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 10 \\ 16 \\ 0 \end{bmatrix}.$$

s_1, s_2 basic x, s_2 basic x, y basic y, s_1 basic

Supplementary problems

1. The set of optimal solutions of any linear programming problem is convex.

Proof. Suppose we have a problem in n decision variables, whose objective function is therefore of the form $z = c^T x$ with $c \in \mathbb{R}^n$, $x \in \mathbb{R}^n$.

The feasible region of the problem is a finite intersection of closed half-spaces, which is convex by theorems 1.1 and 1.3 on pages 80 and 81. Let S denote the feasible region.

If the set of optimal solutions of the problem is empty, then it is convex (the empty set is convex, by the definition on page 79). Otherwise the problem has an optimal solution whose objective value is also optimal. Let k denote the optimal objective value.

The set of optimal solutions is then the intersection of S with the hyperplane $\{x \in \mathbb{R}^n \text{ s.t. } c^T x = k\}$; an intersection of two convex sets which is again convex by theorem 1.3.

Supplementary problems (continued)

2. The set of objective values values which a linear programming problem attains over its feasible region is convex.

Proof Let n, z, c, and S be as in the proof of supplementary problem 1, and recall that the feasible region of any linear programming problem is convex (see paragraph 2, proof of supplementary problem 1).

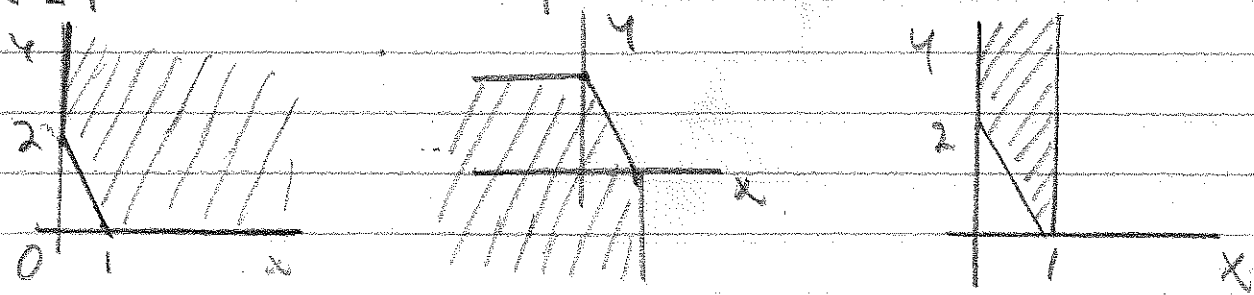
Now pick two objective values, z_1 and z_2, which the linear programming problem takes over its feasible region, and pick $\lambda \in [0,1]$. Then we may find $x_1, x_2 \in S$ such that $z_i = c^T x_i$ $(i = 1, 2)$. Since $\lambda z_1 + (1-\lambda) z_2 = \lambda c^T x_1 + (1-\lambda) c^T x_2$ $= c^T(\lambda x_1 + (1-\lambda) x_2)$, we have expressed the real number $\lambda z_1 + (1-\lambda) z_2$ as the objective value the problem takes at the feasible point $\lambda x_1 + (1-\lambda) x_2$ (using that S is convex).

3. Three examples of convex sets in $\mathbb{R}^2$ which are not line segments and which have $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\2\end{bmatrix}$ as their only extreme points are:

$\left\{\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2 \text{ s.t. } 2x+y \geq 2,\ x \geq 0, \text{ and } y \geq 0\right\}$,

$\left\{\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2 \text{ s.t. } 2x+y \leq 2,\ x \leq 1, \text{ and } y \leq 2\right\}$, and

$\left\{\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2 \text{ s.t. } 2x+y \geq 2, \text{ and } 0 \leq x \leq 1\right\}$.



Supplementary problems (continued)

4. The extreme points of

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \text{ s.t. } x_1 - x_2 + x_3 = -1,\ 3x_1 - 2x_2 + 4x_3 = 2,\ x_1 + x_2 + 3x_3 = 9,\ x_1 \geq 0,\ x_2 \geq 0, \text{ and } x_3 \geq 0 \right\}$$

are $\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$.

To see this note that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ belongs to S only if it is a solution of the system

$$\begin{aligned} x_1 - x_2 + x_3 &= -1 \\ 3x_1 - 2x_2 + 4x_3 &= 2 \\ x_1 + x_2 + 3x_3 &= 9 \end{aligned}.$$

This system has the one-parameter family of solutions $x_1 = 4 - 2x_3$, $x_2 = 5 - x_3$, $x_3 \in \mathbb{R}$.

(using row reduction, $\left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 3 & -2 & 4 & 2 \\ 1 & 1 & 3 & 9 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 5 \\ 0 & 2 & 2 & 10 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$.)

That is, the solution set of the system of equations is a line. The points on this line which satisfy $x_i \geq 0$ for $i = 1, 2, 3$ are precisely those points for which $0 \leq x_3 \leq 2$ (so that $x_1 = 4 - 2x_3 \geq 0$). S is a closed line segment.