APM 236H1F term test 1

12 October, 2011

FAMILY NAME ______________________________

GIVEN NAME(S) ______________________________

STUDENT NUMBER ______________________________

SIGNATURE ______________________________

Instructions: No calculators or other aids allowed.

This test has 3 questions whose values are given immediately after the question numbers. Total marks = 40.

Write solutions in the spaces provided, using the backs of the pages if necessary. (Suggestion: If you have to continue a question, you may use the back of the **previous** page.) Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

The duration of this test is 50 minutes.

1. (13 marks) Write **one linear programming problem** which satisfies **all** of the following:

(i) it has **two decision variables**, x and y

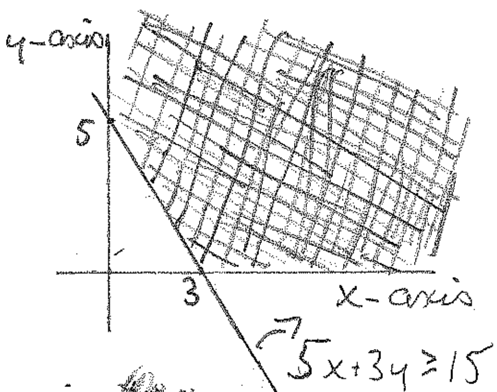
(ii) it is in **standard form**

(iii) it has an **optimal objective value**

(iv) its feasible region is **unbounded**

(v) its feasible region has $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$ as **extreme points** and these are its **only** extreme points.

There are infinitely many correct answers, but none is best. We give 3 solutions below. Graphs were useful in obtaining these.

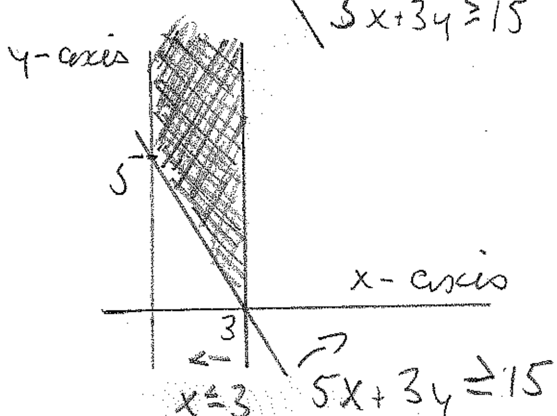


First solution:

Maximize $z = -x - y$ s.t.

$-5x - 3y \leq -15$

$x \geq 0, y \geq 0.$



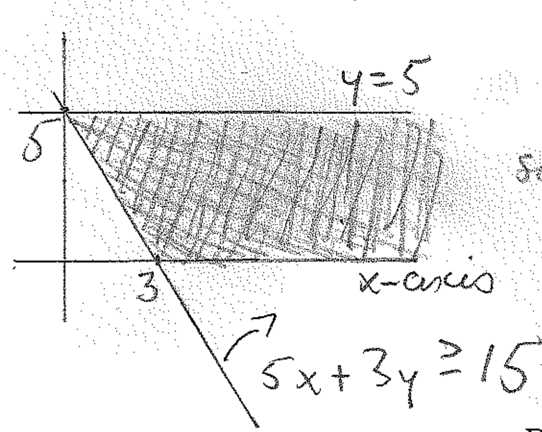
Second solution:

Maximize $z = -y$ s.t.

$-5x - 3y \leq -15$

$x \leq 3$

$x \geq 0, y \geq 0.$



Third solution:

Maximize $z = -x$ s.t.

$-5x - 3y \leq -15$

$y \leq 5$

$x \geq 0, y \geq 0.$

2.(a) (7 marks) In $\mathbb{R}^2$, let S denote the solution set of the non-linear inequality, $xy \le 1$. That is, $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \text{ s.t. } xy \le 1 \right\}$. **Prove** that S is **not convex.**

One correct solution among infinitely many is:

$\begin{bmatrix} 4 \\ 0 \end{bmatrix} \in S$ and $\begin{bmatrix} 0 \\ 4 \end{bmatrix} \in S$ because $4 \cdot 0 = 0 \cdot 4 = 0 \le 1$.

However, the convex combination

$\frac{1}{2}\begin{bmatrix} 4 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \notin S$ because $2 \cdot 2 = 4 \not\le 1$.

2.(b) (7 marks) Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \text{ s.t. } 9x + 5y \le 8 \text{ and } -x + 2y \le 7 \right\}$. **Prove** that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ is **not an extreme point** of S.

One correct solution among infinitely many is given.

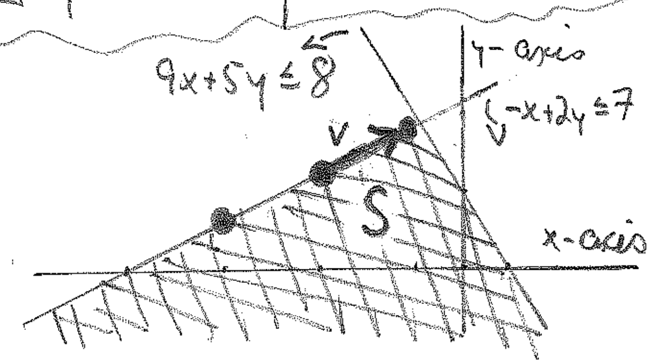
$\begin{bmatrix} -1 \\ 3 \end{bmatrix} \in S$ because $9(-1) + 5(3) = 6 \le 8$, $-(-1) + 2(3) = 7 \le 7$

$\begin{bmatrix} -5 \\ 1 \end{bmatrix} \in S$ because $9(-5) + 5(1) = -40 \le 8$, $-(-5) + 2(1) = 7 \le 7$

Thus $\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -5 \\ 1 \end{bmatrix}$ is expressible as a convex combination of <u>other</u> points of S. Q.E.D.

This diagram was useful in motivating the proof.

$V = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



3. (13 marks) Consider the following linear programming problem (in $\mathbb{R}^4$):

Minimize $z = x_1 + x_2 + x_3 + x_4$ subject to the constraints

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & & & & & = & 4 \\ -3x_1 & + & 6x_2 & + & x_3 & + & 3x_4 & = & -9 \\ & & & & 2x_3 & + & 6x_4 & \geq & 0 \end{array} , \; x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0 .$$

(a) (1 mark) Put the problem in **canonical form**.

(b) (8 marks) Find **all basic solutions** (feasible and infeasible) of the **canonical form** of the problem.

(c) (2 marks) Find **all extreme points** of the feasible region of the problem **given above** (in $\mathbb{R}^4$).

(d) (2 marks) **Solve** the problem **given above** (in $\mathbb{R}^4$). You may assume the problem has an optimal solution.

(a) With x_5 as a slack variable,

Maximize $z = -x_1 - x_2 - x_3 - x_4$ s.t.

$$\begin{aligned} x_1 - 2x_2 &= 4 \\ -3x_1 + 6x_2 + x_3 + 3x_4 &= -9 \\ 2x_3 + 6x_4 - x_5 &= 0 \end{aligned} , \quad x_i \geq 0 \; (i = 1, \dots, 5)$$

(b) The coefficient matrix of the canonical problem is

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ -3 & 6 & 1 & 3 & 0 \\ 0 & 0 & 2 & 6 & -1 \end{bmatrix}.$$

The first and second columns are scalar multiples of each other, as are the third and fourth.

Thus there are only 4 basic solutions:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 6 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 6 \end{bmatrix}.$$

(In each case, the basic variables are the non-zero variables.)

(c) Dropping the infeasible solutions and also the slack variable, the extreme points are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(d) $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is optimal (minimizes the sum of the components).