APM 236H1F term test 1

10 October, 2012

FAMILY NAME ______________________________

GIVEN NAME(S) ______________________________

STUDENT NUMBER ______________________________

SIGNATURE ______________________________

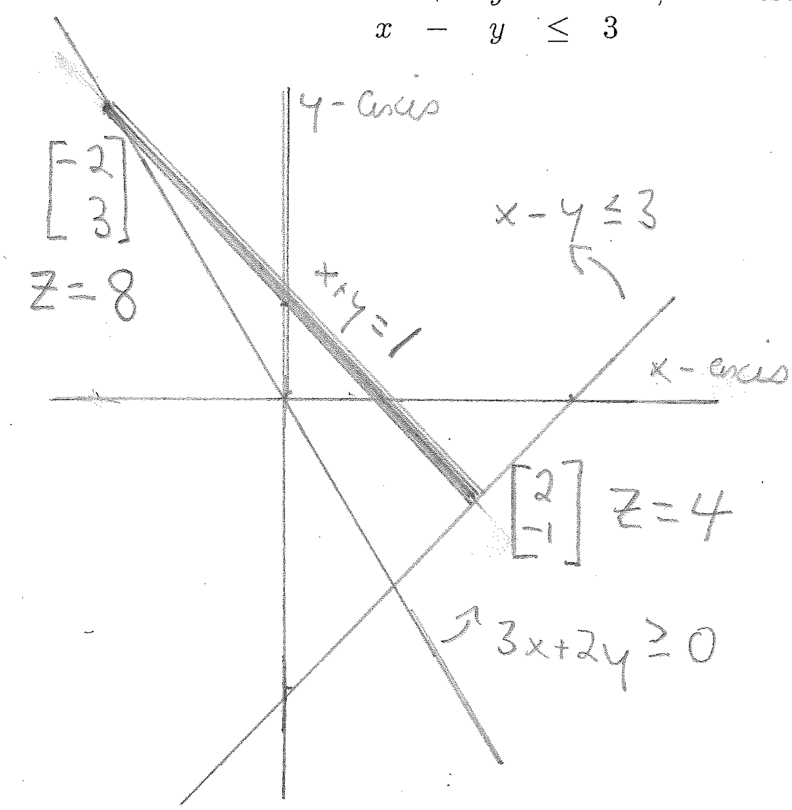
Instructions: No calculators or other aids allowed.

This test has 3 questions whose values are given immediately after the question numbers. Total marks = 40.

Write solutions in the spaces provided, using the backs of the pages if necessary. (Suggestion: If you have to continue a question, you may use the back of the **previous** page.) Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

The duration of this test is 50 minutes.

1. (13 marks) **Solve** the following problem **graphically**: Minimize $z = 5x + 6y$ subject to the constraints
$$\begin{array}{rcrcl} 3x & + & 2y & \geq & 0 \\ x & + & y & = & 1 \\ x & - & y & \leq & 3 \end{array}, \quad x \text{ unrestricted}, \ y \text{ unrestricted}$$



Because of the equality constraint, the feasible region is a (convex) subset of the line $x+y=1$, that is, a line segment.

To find endpoints,

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

In this minimization problem, $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is optimal.

2. (13 marks) A private contractor has three machines which are capable of doing excavation work: a bulldozer, a backhoe, and a crane with clamshell. He has contracted to remove 500 cubic yards (exactly) of material from a certain site, during the week 21 October – 25 October. The number of hours each machine is available that week, the number of cubic yards each machine can remove in one hour, and the hourly operating cost of each machine, are given in the following table:

	availability(hours)	capacity(yards per hour)	cost($ per hour)
bulldozer	30	30	20
backhoe	25	60	27
crane with clamshell	35	40	37

Set up a linear programming problem to determine how many hours during the week each machine should be operated, to complete the job at minimum cost. Having set up the problem, do not solve it.

Let x_1, x_2, x_3 represent, respectively, the number of hours the bulldozer, backhoe, and crane are used.

A linear programming model is:

Minimize $z = 20x_1 + 27x_2 + 37x_3$

subject to the constraints

$$\begin{aligned} 30x_1 + 60x_2 + 40x_3 &= 500 \\ x_1 &\le 30 \\ x_2 &\le 25 \\ x_3 &\le 35 \end{aligned}$$

$$x_1 \ge 0, \; x_2 \ge 0, \; x_3 \ge 0.$$

3. (14 marks) Consider the problem: Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5$ subject to the constraints

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & + & x_3 & & & & & = & 9 \\ & & x_2 & + & x_3 & & & & & = & 5 \\ & & & & x_3 & - & x_4 & + & x_5 & = & 3 \end{array}$$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$.

a) (11 marks) Find **all basic solutions** of the problem's equality constraints.

b) (3 marks) Solve the problem. You may assume that the problem has a solution.

a) In the coefficient matrix
$$\begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

(having columns A_1, A_2, A_3, A_4, A_5 in $\mathbb{R}^3$), any 3 of the last 4 columns are linearly dependent, so x_1 is a basic variable in any basic solution. $\{A_1, A_4, A_5\}$ is also linearly dependent. Thus, there are only 5 basic solutions:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 5 \\ 2 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 4 \\ 0 \\ 5 \\ 0 \\ -2 \end{bmatrix}.$$

(In each solution, the basic variables are the non-zero variables.)

b) Removing the infeasible variables from the above list, the basic feasible solutions are $\begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 0 \\ 0 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 0 \\ 5 \\ 2 \\ 0 \end{bmatrix}$.

$z = 9 \qquad z = 12 \qquad z = 11$

By comparing objective values, $\begin{bmatrix} 4 \\ 2 \\ 3 \\ 0 \\ 0 \end{bmatrix}$ is optimal.