APM 236H1F term test 1

17 October, 2007

FAMILY NAME ______________________________

GIVEN NAME(S) ______________________________

STUDENT NUMBER ______________________________

SIGNATURE ______________________________

Instructions: No calculators or other aids allowed.

This test has 3 questions whose values are given immediately after the question numbers. Total marks = 40.

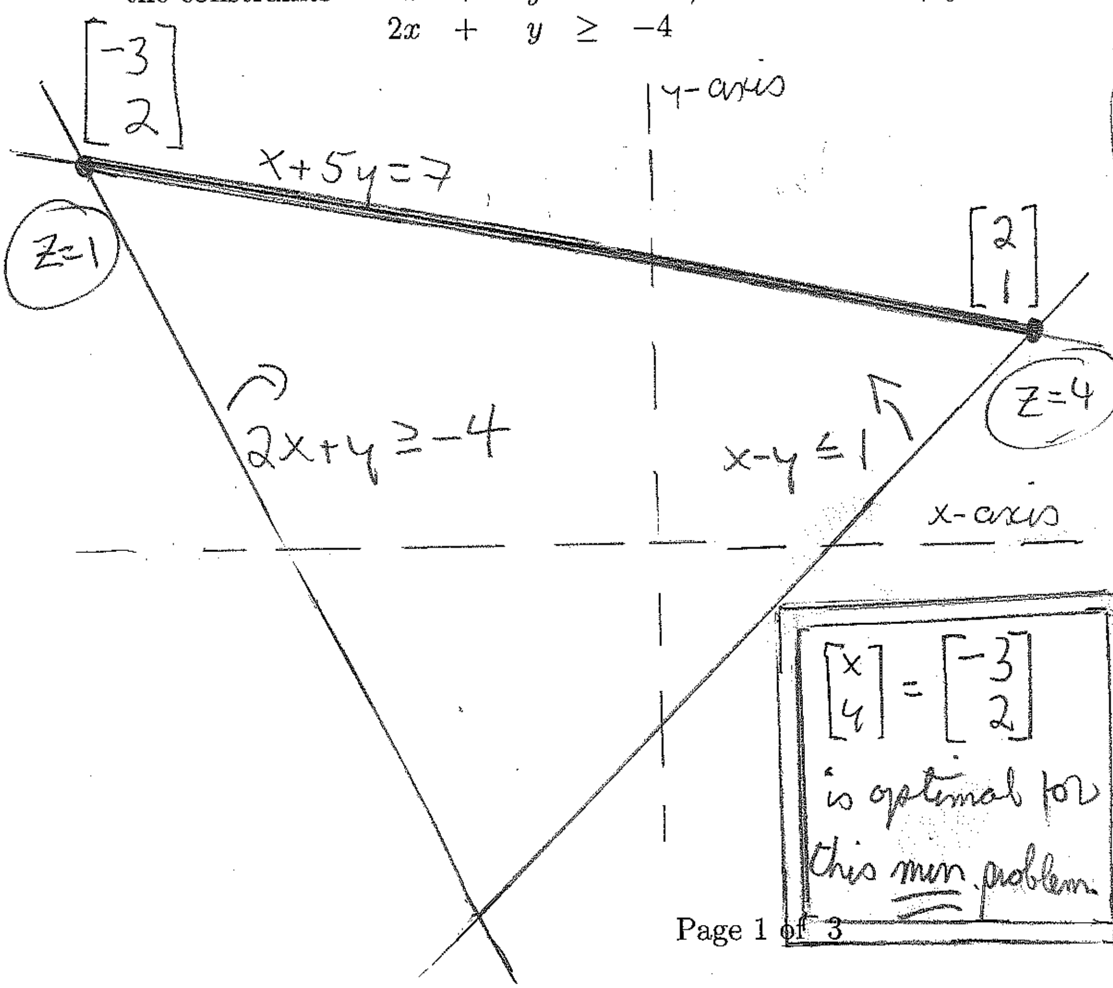
Write solutions in the spaces provided, using the backs of the pages if necessary. (Suggestion: If you have to continue a question, you may use the back of the **previous** page.) Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

The duration of this test is 50 minutes.

1. (13 marks) **Solve** the following problem **graphically**: Minimize $z = x + 2y$ subject to the constraints

$$\begin{array}{rcrcr} x & - & y & \le & 1 \\ x & + & 5y & = & 7 \\ 2x & + & y & \ge & -4 \end{array}$$

, x unrestricted , y unrestricted.



$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}}$$

is optimal for this min problem.

Because of the equality constraint the entire feasible region lies on a line. Endpoints by row reduction:

$$\left[\begin{array}{cc|c} 1 & 5 & 7 \\ 2 & 1 & -4 \end{array}\right] \simeq \left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & -9 & -18 \end{array}\right] \simeq \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 5 & 7 \\ 1 & -1 & 1 \end{array}\right] \simeq \left[\begin{array}{cc|c} 1 & 5 & 7 \\ 0 & -6 & -6 \end{array}\right] \simeq \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

2. (14 marks) **ANSWER ONLY ONE OF THE FOLLOWING PART-QUESTIONS.**

(a) In $\mathbb{R}^2$, **prove** that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is **not** a convex combination of the points $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

(b) Let **S** denote the line segment joining $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ in $\mathbb{R}^2$. Express $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ as a convex combination of $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and a point in **S**.

(a) would be true if, in each solution of the system,

$$c_1 + c_2 + c_3 = 1$$

$$c_1 \begin{bmatrix} 0 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

at least one of c_1, c_2, c_3 were negative.

Solution of the system by row-reduction:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 4 & -1 & 1 \\ 5 & 1 & 0 & 3 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 4 & -1 & 1 \\ 0 & -4 & -5 & -2 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{4} & \frac{3}{4} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & -6 & -1 \end{array}\right] \simeq \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{13}{24} \\ 0 & 1 & 0 & \frac{7}{24} \\ 0 & 0 & 1 & \frac{1}{6} \end{array}\right]$$

The only solution is $c_1 = \frac{13}{24}$, $c_2 = \frac{7}{24}$, $c_3 = \frac{1}{6}$, where all components are positive, which shows (a) is false.

However, $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{13}{24}\begin{bmatrix} 0 \\ 5 \end{bmatrix} + \frac{7}{24}\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ expresses $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ as a convex combination of the 3 points given, and

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{5}{6}\underbrace{\left(\frac{13}{20}\begin{bmatrix} 0 \\ 5 \end{bmatrix} + \frac{7}{20}\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right)}_{\text{in } S} + \frac{1}{6}\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

expresses $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ as a convex combination of $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and a point in S.

3. (13 marks) Consider the following linear programming problem (in $\mathbb{R}^4$):

Minimize $z = x_1 + x_2 + x_3 - x_4$ subject to the constraints

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & - & x_3 & + & 2x_4 & \leq & 6 \\ -x_1 & + & 2x_2 & + & x_3 & & & = & 4 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.$$

(a) (1 mark) Put the problem in **canonical form.**
(b) (8 marks) Find **all basic solutions** (feasible and infeasible) of the **canonical form** of the problem.
(c) (2 marks) Find **all extreme points** of the feasible region of the problem **given above** (in $\mathbb{R}^4$).
(d) (2 marks) **Solve** the problem **given above** (in $\mathbb{R}^4$). You may assume the problem has an optimal solution.

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(a) Maximize $z = -x_1 - x_2 - x_3 + x_4$ s.t.

$$x_1 - 2x_2 - x_3 + 2x_4 + x_5 = 6$$
$$-x_1 + 2x_2 + x_3 = 4$$
$, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0.$

(b) The equality constraints have coefficient matrix

$$\begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array}$$
$$\begin{bmatrix} 1 & -2 & -1 & 2 & 1 \\ -1 & 2 & 1 & 0 & 0 \end{bmatrix}$$

in which $\{A_4, A_5\}$ is linearly dependent, as are any 2 of $\{A_1, A_2, A_3\}$. Thus there are only 6 basic solutions. They are:

$$\begin{bmatrix} -4 \\ 0 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 10 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \\ 10 \end{bmatrix}.$$

(In each case, the basic variables are the non-zero variables.)

(c) Discarding the basic infeasible solutions and dropping x_5,

$$\begin{bmatrix} 0 \\ 2 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$ are extreme.

(d) The z values corresponding to the vectors in (c) are $-3, -1, 2,$ and 4.

$$\begin{bmatrix} 0 \\ 2 \\ 0 \\ 5 \end{bmatrix}$$ is optimal.