APM 236H1F term test 1

14 October, 2009

FAMILY NAME ______________________

GIVEN NAME(S) ______________________

STUDENT NUMBER ______________________

SIGNATURE ______________________

Instructions: No calculators or other aids allowed.

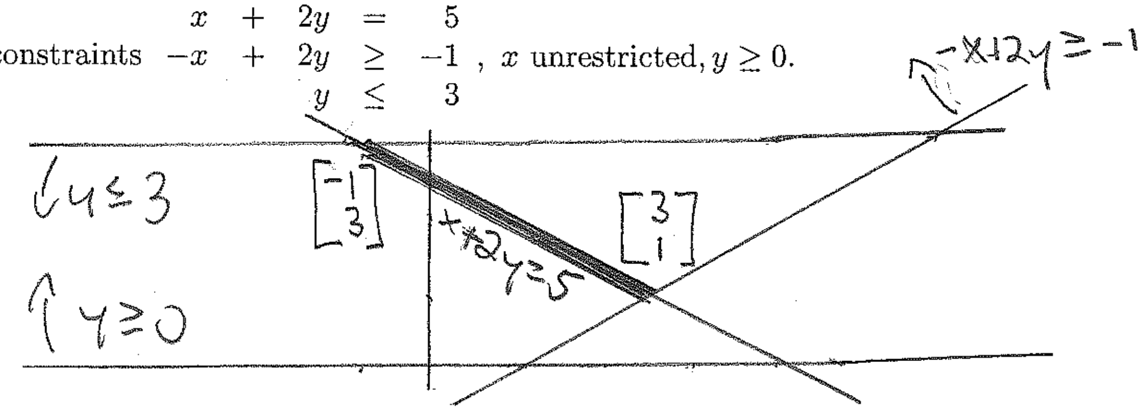
This test has 3 questions whose values are given immediately after the question numbers. Total marks = 40.

Write solutions in the spaces provided, using the backs of the pages if necessary. (Suggestion: If you have to continue a question, you may use the back of the **previous** page.) Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

The duration of this test is 50 minutes.

1. (13 marks) Solve the following problem **graphically**. Minimize $z = 4x + 7y$ subject to the constraints

$$\begin{array}{rcrcr} x & + & 2y & = & 5 \\ -x & + & 2y & \geq & -1 \\ & & y & \leq & 3 \end{array}, \quad x \text{ unrestricted}, y \geq 0.$$



The feasible region lies entirely on the line $x+2y=5$. It is a line segment whose endpoints are the solutions of the system $\begin{matrix} x+2y=5 \\ y=3 \end{matrix}$ and the system $\begin{matrix} x+2y=5 \\ -x+2y=-1 \end{matrix}$. Respective objective values are 17 and 19. This is a <u>minimization</u> problem with solution $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

2. (13 marks) Mr. F. Wizard has decided to invest in three securities—a common stock paying an annual dividend of 8%, a preferred stock paying an annual dividend of 5%, and a bond paying an annual dividend of 3%. He cannot invest more than $10,000 in these securities, but may invest less than $10,000. Also, he has decided that the total amount he will invest in stocks (common and preferred) will not exceed twice the amount he will invest in the bond. Moreover, the amount he will invest in the preferred stock will be at least 30% of the total amount invested in all three securities.

Write a **linear programming problem in standard form** which will determine how much money Mr. Wizard should invest in each security to maximize his total annual return, while following the guidelines he has laid out for himself.

Having formulated the problem, do not solve it.

Let $\$x_1$ = amount invested in common stock
$\$x_2$ = amount invested in preferred stock
$\$x_3$ = amount invested in the bond

Total annual return ($) is $.08x_1 + .05x_2 + .03x_3$

Constraints: $x_1 + x_2 + x_3 \leq 10{,}000$

$$x_1 + x_2 \leq 2x_3$$

$$x_2 \geq .3(x_1 + x_2 + x_3)$$

In standard form, a suitable linear programming model is: Maximize $z = .08x_1 + .05x_2 + .03x_3$

subject to the constraints

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 10{,}000 \\ x_1 + x_2 - 2x_3 &\leq 0 \\ .3x_1 - .7x_2 + .3x_3 &\leq 0 \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 &\geq 0 \end{aligned}$$

3. (14 marks) Consider the linear programming problem: Minimize $z = 3x_1 + x_2 + 2x_3 + 7x_4$ subject to the constraints

$$\begin{array}{rcrcrcrcr} x_1 & & & - & 4x_3 & + & 3x_4 & = & 5 \\ 2x_1 & - & x_2 & - & 8x_3 & + & 6x_4 & = & 10 \end{array},$$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.$

3.(a) (10 marks) Find **all basic solutions** of the system of equality constraints of the problem.

Any two of the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -4 \\ -8 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ form a linearly dependent set so each basic solution has x_2 as a basic variable. The basic solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{(basic variables: } \{x_1, x_2\}\text{)},$$

$$\begin{bmatrix} 0 \\ 0 \\ -\frac{5}{4} \\ 0 \end{bmatrix} \left(\text{basic variables: } \{x_2, x_3\}\right), \text{ and } \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{5}{3} \end{bmatrix} \left(\text{basic variables: } \{x_2, x_4\}\right).$$

3.(b) (4 marks) **Solve** the problem. You may assume that the problem has a solution.

The only basic feasible solutions are $\begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{5}{3} \end{bmatrix}$, with respective objective values 15 and $\frac{35}{3}$. In this minimization problem, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{5}{3} \end{bmatrix}$ is optimal.