APM 236H1F term test 2

15 November, 2006

FAMILY NAME ____

GIVEN NAME(S) ____

STUDENT NUMBER ____

SIGNATURE ____

Instructions: No calculators or other aids allowed.

This test has 3 questions whose values are given immediately after the question numbers. Total marks = 40.

Write solutions in the spaces provided, using the backs of the pages if necessary. (Suggestion: If you have to continue a question, you may use the back of the **previous** page.) Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

The duration of this test is 50 minutes.

1.(a) (7 marks) **Find an optimal solution** of the problem: Maximize $z = 4x_1 + 5x_2 + 2x_3$

subject to the constraints $\begin{array}{rcrcrcr} x_1 & - & x_2 & - & 4x_3 & \geq & -1 \\ 2x_1 & + & x_2 & - & 2x_3 & \leq & 4 \end{array}$, $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

1.(b) (3 marks) **Find a second optimal solution** of the problem of question 1.(a).
1.(c) (3 marks) **Find all optimal solutions** of the problem of question 1.(a).

1.(a) An equivalent canonical problem with slacks x_4, x_5 is:

Maximize $z = 4x_1 + 5x_2 + 2x_3$ s.t. $-x_1 + x_2 + 4x_3 + x_4 = 1$, $2x_1 + x_2 - 2x_3 + x_5 = 4$, $x_i \geq 0$ for $i = 1, \ldots, 5$

Tableau ①

	x_1	x_2	x_3	x_4	x_5	
x_4	-1	1	4	1	0	1
x_5	2	1	-2	0	1	4
	-4	-5	-2	0	0	0

Tableau ②

	x_1	x_2	x_3	x_4	x_4	
x_2	-1	1	4	1	0	1
x_5	3	0	-6	-1	1	3
	-9	0	18	5	0	5

Tableau ③

	x_1	x_2	x_3	x_4	x_5	
x_2	0	1	2	$\frac{2}{3}$	$\frac{1}{3}$	2
x_1	1	0	-2	$-\frac{1}{3}$	$\frac{1}{3}$	1
	0	0	0	2	3	14

$[x_1\ x_2\ x_3]^T = [1\ 2\ 0]^T$ is optimal for the problem (which has 3 decision variables).

1.(b) Noting that if x_3 enters, the objective row will not change, and exiting x_2, we get Tableau ④:

	x_1	x_2	x_3	x_4	x_5	
x_3	0	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{6}$	1
x_1	1	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	3
	0	0	0	2	3	14

$[3\ 0\ 1]^T$ is also optimal.

1.(c) Tableaux ③ and ④ have the only optimal choices of basic variables. The optimal solutions consist of the line segment joining $[1\ 2\ 0]^T$ and $[3\ 0\ 1]^T$. In parametrized form, the line segment is $\left\{ \begin{bmatrix} 1+2\lambda \\ 2-2\lambda \\ \lambda \end{bmatrix} \in \mathbb{R}^3 \text{ s.t. } 0 \leq \lambda \leq 1 \right\} = \left\{ \begin{bmatrix} 3-2\lambda \\ 2\lambda \\ 1-\lambda \end{bmatrix} \in \mathbb{R}^3 \text{ s.t. } 0 \leq \lambda \leq 1 \right\}$.

2. (13 marks) Suppose that, in solving a linear programming problem by the simplex method, we encounter the following non-optimal tableau.

	x_1	$\cdots$	x_j	$\cdots$	x_n	
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_i	a_{i1}	$\cdots$	a_{ij}	$\cdots$	a_{in}	b_i
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
	p_1	$\cdots$	p_j	$\cdots$	p_n	q

Suppose further, that by by following the rules of the simplex method, we enter x_j and exit x_i , but in doing so, the objective value does not change. **Prove** that the basic solution given by the above tableau is degenerate.

Entering x_j and exiting x_i requires that $a_{ij} \neq 0$, and the row-pivot on a_{ij} replaces the objective row with objective row $- p_j (a_{ij}^{-1} \cdot x_i\text{-row})$. The change in objective value will then be $-p_j a_{ij}^{-1} b_i$. Since the tableau is not optimal and we are following the rules of the simplex method, $p_j < 0$.

In particular, $p_j \neq 0$; also $a_{ij}^{-1} \neq 0$. By hypothesis, the change in objective value is 0, so b_i (the value of the basic variable x_i) equals 0.

3. (14 marks) **Solve** the problem: Maximize $z = -4x_1 + x_2 - x_3$ subject to the constraints

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & + & x_3 & \geq & 5 \\ 2x_1 & - & 2x_2 & + & x_3 & = & 2 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Phase 1: x_4 is slack and y_1, y_2 are artificial.

Tableau 1

	x_1	x_2	x_3	x_4	y_1	y_2	
y_1	2	1	1	-1	1	0	5
y_2	2	-2	1	0	0	1	2
	-4	1	-2	1	0	0	-7

Tableau 2

	x_1	x_2	x_3	x_4	y_1	y_2	
y_1	0	3	0	-1	1	-1	3
x_1	1	-1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1
	0	-3	0	1	0	2	-3

Tableau 3

	x_1	x_2	x_3	x_4	y_1	y_2	
x_2	0	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	1
x_1	1	0	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	2
	0	0	0	0	1	1	0

Phase 2, tableau 1

	x_1	x_2	x_3	x_4	
x_2	0	1	0	$-\frac{1}{3}$	1
x_1	1	0	$\frac{1}{2}$	$-\frac{1}{3}$	2
	0	0	-1	1	-7

Phase 2, tableau 2

	x_1	x_2	x_3	x_4	
x_2	0	1	0	$-\frac{1}{3}$	1
x_3	2	0	1	$-\frac{2}{3}$	4
	2	0	0	$\frac{1}{3}$	-3

optimal tableau