APM 236H1F term test 2

14 November, 2007

FAMILY NAME ______________________________

GIVEN NAME(S) ______________________________

STUDENT NUMBER ______________________________

SIGNATURE ______________________________

Instructions: No calculators or other aids allowed.

This test has 3 questions whose values are given immediately after the question numbers. Total marks = 40.

Write solutions in the spaces provided, using the backs of the pages if necessary. (Suggestion: If you have to continue a question, you may use the back of the **previous** page.) Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

The duration of this test is 50 minutes.

1.(a) (7 marks) Find **an optimal solution** of the problem: Maximize $z = x_1 - 2x_2 - 2x_3$

subject to the constraints $\begin{array}{rcrcrcl} x_1 & - & 2x_2 & - & x_3 & \leq & 2 \\ -x_1 & + & x_2 & + & x_3 & \leq & 1 \end{array}$, $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

Tableau 1

	x_1	x_2	x_3	x_4	x_5	
x_4	1	-2	-1	1	0	2
x_5	-1	1	1	0	1	1
	-1	2	2	0	0	0

Tableau 2

	x_1	x_2	x_3	x_4	x_5	
x_1	1	-2	-1	1	0	2
x_5	0	-1	0	1	1	3
	0	0	1	1	0	2

1.(b) (7 marks) Find **all optimal solutions** of the problem in question 1.(a).

The objective row coefficient of x_2 ("0") indicates that if x_2 increases, the objective value will not change. The non-positive coefficients of x_2 in the x_1 and x_5 rows indicate that an increase in x_2 feasible, provided there is a corresponding increase in x_1 and x_5. The set of optimal solutions is

$$\left\{ \begin{bmatrix} 2+2M \\ M \\ 0 \\ 0 \\ 3+M \end{bmatrix} \in \mathbb{R}^5 \text{ s.t. } M \geq 0 \right\}$$ or (for the given standard problem) $$\left\{ \begin{bmatrix} 2+2M \\ M \\ 0 \end{bmatrix} \in \mathbb{R}^3 \text{ s.t. } M \geq 0 \right\}.$$

2. (13 marks) Suppose in solving a linear programming problem by the simplex method we encounter a tableau, part of which is given below, where $a_1 > 0$ and $a_m > 0$.

	x_j	
x_1	a_1	b_1
$\vdots$	$\vdots$	$\vdots$
x_m	a_m	b_m
	-1	0

In the next iteration of the simplex method, x_j will enter. Now suppose that the θ ratio for the x_m row is less than the θ ratio for the x_1 row and, contrary to the rules of the simplex method, we exit x_1. **Prove** that the next tableau will be infeasible.

Entering x_j and exiting x_1 will produce a tableau where x_m has the value

$$b_m - a_m \frac{b_1}{a_1} = a_m \left(\frac{b_m}{a_m} - \frac{b_1}{a_1} \right).$$

Since the θ ratios $\frac{b_m}{a_m}$ and $\frac{b_1}{a_1}$ satisfy

$\frac{b_m}{a_m} < \frac{b_1}{a_1}$, we have $\frac{b_m}{a_m} - \frac{b_1}{a_1} < 0$

(while $a_m > 0$), so that the value of x_m will be <u>negative</u>.

3. (13 marks) **Solve** the problem: Maximize $z = x_1 - x_2 - 4x_3$

subject to the constraints
$$\begin{array}{rcrcrcr} x_1 & - & x_2 & - & x_3 & \le & -3 \\ x_1 & + & x_2 & + & 2x_3 & = & 9 \\ x_1 & - & x_2 & + & x_3 & \le & 6 \end{array}, \quad x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 .$$

Phase 1, tableau 1

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	
y_1	-1	1	1	-1	0	1	0	3
y_2	1	1	2	0	0	0	1	9
x_5	1	-1	1	0	1	0	0	6
	0	-2	-3	1	0	0	0	-12

Phase 1, tableau 2

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	
x_3	-1	1	1	-1	0	1	0	3
y_2	3	-1	0	2	0	-2	1	3
x_5	2	-2	0	1	1	-1	0	3
	-3	1	0	-2	0	3	0	-3

Phase 1, tableau 3

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	
x_3	0	$\frac{2}{3}$	1	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	4
x_1	1	$-\frac{1}{3}$	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{1}{3}$	1
x_5	0	$-\frac{4}{3}$	0	$-\frac{1}{3}$	1	$\frac{1}{3}$	$-\frac{2}{3}$	1
	0	0	0	0	0	1	1	0

Phase 2, tableau 1

	x_1	x_2	x_3	x_4	x_5	
x_3	0	$\frac{2}{3}$	1	$-\frac{1}{3}$	0	4
x_1	1	$-\frac{1}{3}$	0	$\frac{2}{3}$	0	1
x_5	0	$-\frac{4}{3}$	0	$-\frac{1}{3}$	1	1
	0	-2	0	2	0	-15

Phase 2, tableau 2

	x_1	x_2	x_3	x_4	x_5	
x_2	0	1	$\frac{3}{2}$	$-\frac{1}{2}$	0	6
x_1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	3
x_5	0	0	2	-1	1	9
	0	0	3	1	0	-3