# APM 236H1F term test 2

12 November, 2008

FAMILY NAME ______________________________

GIVEN NAME(S) ______________________________

STUDENT NUMBER ______________________________

SIGNATURE ______________________________

**Instructions: No calculators or other aids allowed.**

This test has 3 questions whose values are given immediately after the question numbers. Total marks = 40.

Write solutions in the spaces provided, using the backs of the pages if necessary. (Suggestion: If you have to continue a question, you may use the back of the **previous** page.) Aspects of any question which are indicated in **boldface** will be regarded as crucial during grading. **Show your work.**

The duration of this test is 50 minutes.

1. (13 marks) **Solve** the problem: Maximize $z = 7x_1 + 4x_2 + 2x_3$ subject to the constraints

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & - & 4x_3 & \leq & 0 \\ 2x_1 & - & 3x_2 & + & 2x_3 & \geq & 0 \\ x_1 & + & x_2 & + & x_3 & \leq & 10 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Note that the second constraint says "$\geq$". This is **not** a typographical error.

After replacing the constraints with

$$\begin{array}{l} x_1 + x_2 - 4x_3 + x_4 = 0 \\ -2x_1 + 3x_2 - 2x_3 + x_5 = 0 \\ x_1 + x_2 + x_3 + x_6 = 10 \end{array}$$

with slacks $x_4, x_5, x_6$, the simplex sequence which solves the problem is

Tableau 1

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | |
|---|---|---|---|---|---|---|---|
| $x_4$ | 1 | 1 | -4 | 1 | 0 | 0 | 0 |
| $x_5$ | -2 | 3 | -2 | 0 | 1 | 0 | 0 |
| $x_6$ | 1 | 1 | 1 | 0 | 0 | 1 | 10 |
| | -7 | -4 | -2 | 0 | 0 | 0 | 0 |

Tableau 2

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 1 | -4 | 1 | 0 | 0 | 0 |
| $x_5$ | 0 | 5 | -10 | 2 | 1 | 0 | 0 |
| $x_6$ | 0 | 0 | 5 | -1 | 0 | 1 | 10 |
| | 0 | 3 | -30 | 7 | 0 | 0 | 0 |

Tableau 3

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | |
|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 1 | 0 | $\frac{1}{5}$ | 0 | $\frac{4}{5}$ | 8 |
| $x_5$ | 0 | 5 | 0 | 0 | 1 | 2 | 20 |
| $x_3$ | 0 | 0 | 1 | $-\frac{1}{5}$ | 0 | $\frac{1}{5}$ | 2 |
| | 0 | 3 | 0 | 1 | 0 | 6 | 60 |

2. (13 marks) Suppose in solving a linear programming problem by the simplex method we encounter the following non-degenerate, non-optimal tableau.

| | $x_1$ | $\cdots$ | $x_j$ | $\cdots$ | $x_n$ | |
|---|---|---|---|---|---|---|
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ |
| $x_i$ | $a_{i1}$ | $\cdots$ | $a_{ij}$ | $\cdots$ | $a_{in}$ | $b_i$ |
| $\vdots$ | $\vdots$ | | $\vdots$ | | $\vdots$ | $\vdots$ |
| | $p_1$ | $\cdots$ | $p_j$ | $\cdots$ | $p_n$ | $q$ |

Suppose further, that by following the rules of the simplex method, we will enter $x_j$ and exit $x_i$. **Prove** that this will cause the objective value to **increase**.

The row-pivot which enters $x_j$ and exits $x_i$ will replace the objective row with

objective row $- p_j a_{ij}^{-1} \cdot x_i$-row

This will cause the objective value $q$ to be replaced with $q - p_j a_{ij}^{-1} b_i$.

Since the above tableau is non-optimal and we are following the simplex method, $p_j < 0$. Again because we have been following the simplex method, $b_i \geq 0$. Since the tableau is non-degenerate, $b_i \neq 0$ so that $b_i > 0$. $x_j$ will enter with value $a_{ij}^{-1} b_i$ which implies $a_{ij} > 0$ — necessary (because $b_i$ is <u>positive</u>) to ensure the feasibility of the next tableau ($x_j \geq 0$).

Thus $-p_j a_{ij}^{-1} b_i > 0$ and $q - p_j a_{ij}^{-1} b_j > q$.

3. (14 marks) **Solve** the problem: Maximize $z = 2x_2 + x_3$ subject to the constraints

$$\begin{array}{rcrcrcl} -6x_1 & + & 3x_2 & + & x_3 & = & 3 \\ & & 3x_2 & + & 4x_3 & = & 9 \\ 2x_1 & & & + & x_3 & = & 2 \end{array}, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$$

Phase 1, tableau 1

artificial

| | $x_1$ | $x_2$ | $x_3$ | $y_1$ | $y_2$ | $y_3$ | |
|---|---|---|---|---|---|---|---|
| $y_1$ | $-6$ | 3 | 1 | 1 | 0 | 0 | 3 |
| $y_2$ | 0 | 3 | 4 | 0 | 1 | 0 | 9 |
| $y_3$ | 2 | 0 | 1 | 0 | 0 | 1 | 2 |
| | 4 | $-6$ | $-6$ | 0 | 0 | 0 | $-14$ |

Phase 1, tableau 2

| | $x_1$ | $x_2$ | $x_3$ | $y_1$ | $y_2$ | $y_3$ | |
|---|---|---|---|---|---|---|---|
| $y_1$ | $-8$ | 3 | 0 | 1 | 0 | $-1$ | 1 |
| $y_2$ | $-8$ | 3 | 0 | 0 | 1 | $-4$ | 1 |
| $x_3$ | 2 | 0 | 1 | 0 | 0 | 1 | 2 |
| | 16 | $-6$ | 0 | 0 | 0 | 6 | $-2$ |

Phase 1, tableau 3

| | $x_1$ | $x_2$ | $x_3$ | $y_1$ | $y_2$ | $y_3$ | |
|---|---|---|---|---|---|---|---|
| $x_2$ | $-\frac{8}{3}$ | 1 | 0 | $\frac{1}{3}$ | 0 | $-\frac{1}{3}$ | $\frac{1}{3}$ |
| $y_2$ | 0 | 0 | 0 | $-1$ | 1 | $-3$ | 0 |
| $x_3$ | 2 | 0 | 1 | 0 | 0 | 1 | 2 |
| | 0 | 0 | 0 | 2 | 0 | 4 | 0 |

Phase 2, tableau 1

| | $x_1$ | $x_2$ | $x_3$ | $y_2$ | |
|---|---|---|---|---|---|
| $x_2$ | $-\frac{8}{3}$ | 1 | 0 | 0 | $\frac{1}{3}$ |
| $y_2$ | 0 | 0 | 0 | 1 | 0 |
| $x_3$ | 2 | 0 | 1 | 0 | 2 |
| | $-\frac{10}{3}$ | 0 | 0 | 0 | $\frac{8}{3}$ |

Phase 2, tableau 2

| | $x_1$ | $x_2$ | $x_3$ | $y_2$ | |
|---|---|---|---|---|---|
| $x_2$ | 0 | 1 | $\frac{4}{3}$ | 0 | 3 |
| $y_2$ | 0 | 0 | 0 | 1 | 0 |
| $x_1$ | 1 | 0 | $\frac{1}{2}$ | 0 | 1 |
| | 0 | 0 | $\frac{5}{3}$ | 0 | 6 |