CHAPTER I

GROUP AND HAMILTONIAN STRUCTURES OF FLUID DYNAMICS

The group we will most often be dealing with in hydrodynamics is the infinite-dimensional group of diffeomorphisms that preserve the volume element of the domain of a fluid flow. One can also relate many rather interesting systems to other groups, in particular, to finite-dimensional ones. For example, the ordinary theory of a rigid body with a fixed point corresponds to the rotation group $SO(3)$, while the Lobachevsky geometry has to do with the group of translations and dilations of a vector space. Our constructions are equally applicable to the gauge groups exploited by physicists. The latter groups occupy an intermediate position between the rotation group of a rigid body and the diffeomorphism groups. They are already infinite-dimensional but yet too simple to serve as a model for hydrodynamics.

In this chapter we study geodesics of one-sided invariant Riemannian metrics on Lie groups. The principle of least action asserts that motions of physical systems such as rigid bodies and ideal fluids are described by the geodesics in these metrics given by the kinetic energy.

§1 Symmetry groups for a rigid body and an ideal fluid

Definition 1.1. A set G of smooth transformations of a manifold M into itself is called a *group* if

i) along with every two transformations $g, h \in G$, the composition $g \circ h$ belongs to G (the symbol $g \circ h$ means that one first applies h and then g);

ii) along with every $g \in G$, the inverse transformation g^{-1} belongs to G as well.

From *i*) and *ii*) it follows that every group contains the identity transformation (the unity) e.

A group is called a *Lie group* if G has a smooth structure and the operations *i*) and *ii*) are smooth.

Example 1.2. All rotations of a rigid body about the origin form the Lie group $SO(3)$.

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EXAMPLE 1.3. Diffeomorphisms preserving the volume element in a domain M form a Lie group. Throughout the book we denote this group by $S\mathrm{Diff}(M)$ (or by $\mathcal{D}$ to avoid complicated formulas).

The group $S\mathrm{Diff}(M)$ can be regarded as the configuration space of an incompressible fluid filling the domain M. Indeed, a fluid flow determines for every time moment t the map g^t of the flow domain to itself (the initial position of every fluid particle is taken to its terminal position at the moment t). All the terminal positions, i.e., configurations of the system (or "permutations of particles"), form the "infinite-dimensional manifold" $S\mathrm{Diff}(M)$. (Here and in the sequel we consider only the diffeomorphisms of M that can be connected with the identity transformation by a continuous family of diffeomorphisms. Our notation $S\mathrm{Diff}(M)$ stands only for the connected component of the identity of the group of all volume-preserving diffeomorphisms of M.)

The kinetic energy of a fluid (under the assumption that the fluid density is 1) is the integral (over the flow domain) of half the square of the velocity of the fluid particles. Since the fluid is incompressible, the integration can be carried out either with the volume element occupied by an initial particle or with the volume element dx occupied by that at the moment t:

$$E = \frac{1}{2}\int_M v^2\ dx,$$

where v is the velocity field of the fluid: $v(x,t) = \frac{\partial}{\partial t}g^t(y)$, $x = g^t(y)$ (y is an initial position of the particle whose position is x at the moment t), see Fig.1.

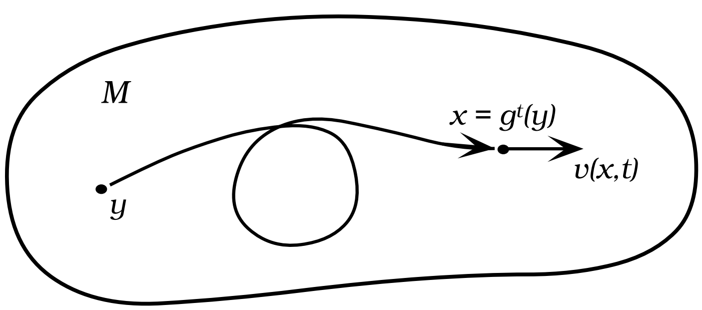

FIGURE 1. The motion of a fluid particle in a domain M.

Suppose that a configuration g changes with velocity $\dot{g}$. The vector $\dot{g}$ belongs to the tangent space T_gG of the group $G = S\mathrm{Diff}(M)$ at the point g. The kinetic energy is a quadratic form on this vector space of velocities.

THEOREM 1.4. *The kinetic energy of an incompressible fluid is invariant with respect to the* right *translations on the group $G = S\,\mathrm{Diff}(M)$ (i.e., with respect to the mappings $R_h : G \to G$ of the type $R_h(g) = gh$).*

PROOF. The multiplication of all group elements by h from the *right* means that the diffeomorphism h (preserving the volume element) acts *first*, before a diffeomorphism g changing with the velocity $\dot{g}$. Such a diffeomorphism h can be regarded as a (volume-preserving) renumeration of particles at the initial position, $y = h(z)$. The velocity of the particle occupying a certain position at a given moment does not change under the renumeration, and therefore the kinetic energy is preserved. □

Similarly, the kinetic energy of a rigid body fixed at some point is a quadratic form on every tangent space to the configuration space of the rigid body, i.e., to the manifold $G = SO(3)$.

THEOREM 1.5. *The kinetic energy of a rigid body is invariant with respect to the* left *translations on the group $G = SO(3)$, i.e., with respect to the transformations $L_h : G \to G$ having the form $L_h(g) = hg$.*

PROOF. The multiplication of the group elements by h from the *left* means that the rotation h is carried out *after* the rotation g, changing with the velocity $\dot{g}$. Such a rotation h can be regarded as a revolution of the entire space, along with the rotating body. This revolution does not change the length of the velocity vector of each point of the body, and hence it does not change the total kinetic energy. □

REMARK 1.6. On the group $SO(3)$ (and more generally, on every compact group) there exists a two-sided invariant metric. On the infinite-dimensional groups of most interest for hydrodynamics, there is no such Riemannian metric. However, for two- and three-dimensional hydrodynamics, on the corresponding groups of volume-preserving diffeomorphisms there are two-sided invariant nondegenerate quadratic forms in every tangent space (see Section IV.8.C for the two-dimensional case, and Sections III.4 and IV.8.D for three dimensions, where this quadratic form is "helicity").

§2. Lie groups, Lie algebras, and adjoint representation

In this section we set forth basic facts about Lie groups and Lie algebras in the form and with the notations used in the sequel.

A linear coordinate change C sends an operator matrix B to the matrix CBC^{-1}. A similar construction exists for an arbitrary Lie group G.

DEFINITION 2.1. The composition $A_g = R_{g^{-1}} L_g : G \to G$ of the right and left translations, which sends any group element $h \in G$ to ghg^{-1}, is called an *inner automorphism* of the group G. (The product of $R_{g^{-1}}$ and L_g can be taken in either order: all the left translations commute with all the right ones.) It is indeed an automorphism, since

$$A_g(fh) = (A_g f)(A_g h).$$

The map sending a group element g to the inner automorphism A_g is a group homomorphism, since $A_{gh} = A_g A_h$.

The inner automorphism A_g does not affect the group unity. Hence, its derivative at the unity takes the tangent space to the group at the unity to itself.

DEFINITION 2.2. The tangent space to the Lie group at the unity is called the *vector space of the Lie algebra* corresponding to the group.

The Lie algebra of a group G is usually denoted by the corresponding Gothic letter $\mathfrak{g}$.

EXAMPLE 2.3. For the Lie group $G = S\mathrm{Diff}(M)$, formed by the diffeomorphisms preserving the volume element of the flow domain M, the corresponding Lie algebra consists of divergence-free vector fields in M.

EXAMPLE 2.4. The Lie algebra $\mathfrak{so}(n)$ of the rotation group $SO(n)$ consists of skew-symmetric $n \times n$ matrices. For $n = 3$ the vector space of skew-symmetric matrices is three-dimensional. The vectors of this three-dimensional space are said to be *angular velocities.*

DEFINITION 2.5. The differential of the inner automorphism A_g at the group unity e is called the *group adjoint operator* Ad_g:

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}, \qquad \mathrm{Ad}_g\, a = (A_{g*}\,|_e)a, \qquad a \in \mathfrak{g} = T_eG.$$

(Here and in the sequel, we denote by T_xM the tangent space of the manifold M at the point x, and by $F_*|_x : T_xM \to T_{F(x)}M$ the derivative of the mapping $F : M \to M$ at x. The derivative F_* of F at x is a linear operator.)

The adjoint operators form a representation of the group: $\mathrm{Ad}_{gh} = \mathrm{Ad}_g \mathrm{Ad}_h$ by the linear operators acting in the Lie algebra space.

EXAMPLE 2.6. The adjoint operators of the group $S\mathrm{Diff}(M)$ define the diffeomorphism action on divergence-free vector fields in M as the coordinate changes in the manifold.

The map Ad, which associates the operator Ad_g to a group element $g \in G$, may be regarded as a map from the group to the space of the linear operators in the Lie algebra.

DEFINITION 2.7. The differential ad of the map Ad at the group unity is called the *adjoint representation of the Lie algebra*:

$$\mathrm{ad} = \mathrm{Ad}_{*e} : \mathfrak{g} \to \mathrm{End}\ \mathfrak{g}, \qquad \mathrm{ad}_\xi = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{g(t)},$$

where $g(t)$ is a curve on the group G issued from the point $g(0) = e$ with the velocity $\dot{g}(0) = \xi$ (Fig.2). Here, End $\mathfrak{g}$ is the space of linear operators taking $\mathfrak{g}$ to itself. The symbol ad_ξ stands for the image of an element ξ, from the Lie algebra $\mathfrak{g}$, under the action of the linear map ad. This image $\mathrm{ad}_\xi \in$ End $\mathfrak{g}$ is itself a linear operator in $\mathfrak{g}$.

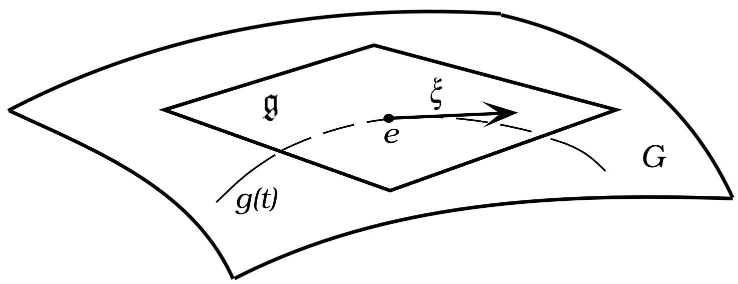

FIGURE 2. The vector ξ in the Lie algebra $\mathfrak{g}$ is the velocity at the identity e of a path $g(t)$ on the Lie group G.

EXAMPLE 2.8. Let G be the rotation group in $\mathbb{R}^n$. Then

$$\mathrm{ad}_\xi\, \omega = [\xi, \omega],$$

where $[\xi, \omega] = \xi\omega - \omega\xi$ is the commutator of skew-symmetric matrices ξ and ω. In particular, for $n = 3$ the vector $[\xi, \omega]$ is the ordinary cross product $\xi \times \omega$ of the angular velocity vectors ξ and ω in $\mathbb{R}^3$.

PROOF. Let $t \mapsto g(t)$ be a curve issuing from e with the initial velocity $\dot{g} = \xi$, and let $s \mapsto h(s)$ be such a curve with the initial velocity $h' = \omega$. Then

$$\begin{aligned} g(t)h(s)g(t)^{-1} &= (e + t\xi + o(t))(e + s\omega + o(s))(e + t\xi + o(t))^{-1} \\ &= e + s[\omega + t(\xi\omega - \omega\xi) + o(t)] + o(s) \end{aligned}$$

as $t, s \to 0$. □

EXAMPLE 2.9. Let $G =$ Diff(M) be the group of diffeomorphisms of a manifold M. Then

$$\mathrm{ad}_v\, w = -\{v, w\}, \tag{2.1}$$

where $\{v, w\}$ is the Poisson bracket of vector fields v and w.

The *Poisson bracket of vector fields* is defined as the commutator of the corresponding differential operators:

$$L_{\{v,w\}} = L_v L_w - L_w L_v. \tag{2.2}$$

The linear first-order differential operator L_v, associated to a vector field v, is the derivative along the vector field v ($L_v f = \sum v_i \frac{\partial f}{\partial x_i}$ for an arbitrary function f and any coordinate system).

The components of the field $\{v, w\}$ in an arbitrary coordinate system are expressed in terms of the components of w and v according to the following formula:

$$\{v, w\}_i = \sum_j v_j \frac{\partial w_i}{\partial x_j} - w_j \frac{\partial v_i}{\partial x_j}.$$

It follows from the above that the field $\{v, w\}$ does not depend on the coordinate system $(x_1, \ldots, x_n)$ used in the latter formula.

The operator L_v (called the *Lie derivative*) also acts on any tensor field on a manifold, and it is defined as the "fisherman derivative": the flow is transporting the tensors in front of the fisherman, who is sitting at a fixed place and differentiates in time what he sees. For instance, the functions are transported backwards by the flow, and hence $L_v f = \sum v_i \frac{\partial f}{\partial x_i}$. Similarly, differential forms are transported backwards, but vector fields are transported forwards. Thus, for vector fields we obtain that $L_v w = -\{v, w\}$.

The minus sign enters formula (2.1) since, traditionally, the sign of the Poisson bracket of two vector fields is defined according to (2.2), similar to the matrix commutator. The opposite signs in the last two examples result from the same reason as the distinction in invariance of the kinetic energy: It is left invariant in the former case and right invariant in the latter.

Proof of Formula (2.1). Diffeomorphisms corresponding to the vector fields v and w can be written (in local coordinates) in the form

$$g(t) : x \mapsto x + tv(x) + o(t)\ ,\ t \to 0,$$
$$h(s) : x \mapsto x + sw(x) + o(s)\ ,\ s \to 0.$$

Then we have $g(t)^{-1} : x \mapsto x - tv(x) + o(t)$, whence

$$\begin{aligned} h(s)(g(t))^{-1} :\ x \mapsto\ & x - tv(x) + o(t) + sw(x - tv(x) + o(t)) + o(s) \\ &= x - tv(x) + o(t) + s\left(w(x) - t\frac{\partial w}{\partial x}v(x) + o(t)\right) + o(s), \text{ and} \end{aligned}$$

$$g(t)h(s)(g(t))^{-1} : x \mapsto x + s\left(w(x) + t\left(\frac{\partial v}{\partial x}w(x) - \frac{\partial w}{\partial x}v(x)\right)\right) + o(t) + o(s).$$

□

EXAMPLE 2.10. Let $G = S\mathrm{Diff}(M)$ be the group of diffeomorphisms preserving the volume element in a domain M. Formula (2.1) is valid in this case, while all the three fields v, w, and $\{v, w\}$ are divergence free.

DEFINITION 2.11. The *commutator* in the Lie algebra $\mathfrak{g}$ is defined as the operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that associates to a pair of vectors a, b of the tangent space $\mathfrak{g}$ (at the unity of a Lie group G) the following third vector of this space:

$$[a, b] = \mathrm{ad}_a\, b.$$

The tangent space at the unity of the Lie group equipped with such operation $[\ ,\ ]$ is called the *Lie algebra of the Lie group* G.

EXAMPLE 2.12. The commutator of skew-symmetric matrices a and b is $ab - ba$ (in the three-dimensional case it is the cross product $a \times b$ of the corresponding vectors). The commutator of two vector fields is minus their Poisson bracket. The commutator of divergence-free vector fields in a three-dimensional Euclidean space is given by the formula

$$[a, b] = \ \mathrm{curl}\ (a \times b),$$

where $a \times b$ is the cross product. It follows from the more general formula

$$\mathrm{curl}\ (a \times b) = [a, b] + a\ \mathrm{div}\ b - b\ \mathrm{div}\ a,$$

and it is valid for an arbitrary Riemannian three-dimensional manifold M^3. The latter formula may be obtained by the repeated application of the homotopy formula (see Section 7.B).

REMARK 2.13. The commutation operation in any Lie algebra can be defined by the following construction. Extend the vectors v and w in the left-invariant way to the entire Lie group G. In other words, at every point $g \in G$, we define a tangent vector $v_g \in T_g G$, which is the left translation by g of the vector $v \in \mathfrak{g} = T_e G$. We obtain two *left-invariant* vector fields $\tilde{v}$ and $\tilde{w}$ on G. Take their Poisson bracket $\tilde{u} = \{\tilde{v}, \tilde{w}\}$. The Poisson bracket operation is invariant under the diffeomorphisms. Hence the field $\tilde{u}$ is also *left*-invariant, and it is completely determined by its value u at the group unity. The latter vector $u \in T_e G = \mathfrak{g}$ can be taken as the definition of the commutator in the Lie algebra $\mathfrak{g}$:

$$[v, w] = u.$$

The analogous construction carried out with *right-invariant* fields $\tilde{v}, \tilde{w}$ on the group G provides us with *minus* the commutator.

THEOREM 2.14. *The commutator operation* $[\ ,\ ]$ *is bilinear, skew-symmetric, and satisfies the Jacobi identity:*

$$\begin{aligned}
&[\lambda a + \nu b, c] = \lambda [a, c] + \nu [b, c]; \\
&[a, b] = -[b, a]; \\
&[[a, b], c] + [[b, c], a] + [[c, a], b] = 0.
\end{aligned}$$

REMARK 2.15. A vector space equipped with a bilinear skew-symmetric operation satisfying the Jacobi identity is called an *abstract Lie algebra.* Every (finite-dimensional) abstract Lie algebra is the Lie algebra of a certain Lie group G.

Unfortunately, in the infinite-dimensional case this is not so. This is a source of many difficulties in quantum field theory, in the theory of completely integrable systems, and in other areas where the language of infinite-dimensional Lie algebras replaces that of Lie groups (see, e.g., Section VI.1 on the Virasoro algebra and KdV equation). One can view a Lie algebra as the first approximation to a Lie group, and the Jacobi identity appears as the infinitesimal consequence of associativity of the group multiplication. In a finite-dimensional situation a (connected simply connected) Lie group itself can be reconstructed from its first approximation. However, in the infinite-dimensional case such an attempt at reconstruction may lead to divergent series.

It is easy to verify the following

THEOREM 2.16. *The adjoint operators* $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ *form a representation of a Lie group* G *by the automorphisms of its Lie algebra* $\mathfrak{g}$*:*

$$[\mathrm{Ad}_g \xi, \mathrm{Ad}_g \eta] = \mathrm{Ad}_g [\xi, \eta], \qquad \mathrm{Ad}_{gh} = \mathrm{Ad}_g \mathrm{Ad}_h .$$

DEFINITION 2.17. The set of images of a Lie algebra element ξ, under the action of all the operators Ad_g, $g \in G$, is called the *adjoint* (group) *orbit* of ξ.

EXAMPLES 2.18. A) The adjoint orbit of a matrix, regarded as an element of the Lie algebra of all complex matrices, is the set of matrices with the same Jordan normal form.

B) The adjoint orbits of the rotation group of a three-dimensional Euclidean space are spheres centered at the origin, and the origin itself.

C) The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of the group $SL(2, \mathbb{R})$ of real matrices with the unit determinant consists of all traceless 2×2 matrices:

$$\mathfrak{sl}(2, \mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}$$

with real a, b, and c. Matrices with the same Jordan normal form have equal values of the determinant $\Delta = -(a^2 + bc)$. The adjoint orbits in $\mathfrak{sl}(2, \mathbb{R})$ are defined by this determinant "almost uniquely," though they are finer than in the complex case. The orbits are the connected components of the quadrics $a^2 + bc = \text{const} \neq 0$, each half of the cone $a^2 + bc = 0$, and the origin $a = b = c = 0$, see Fig.3a.

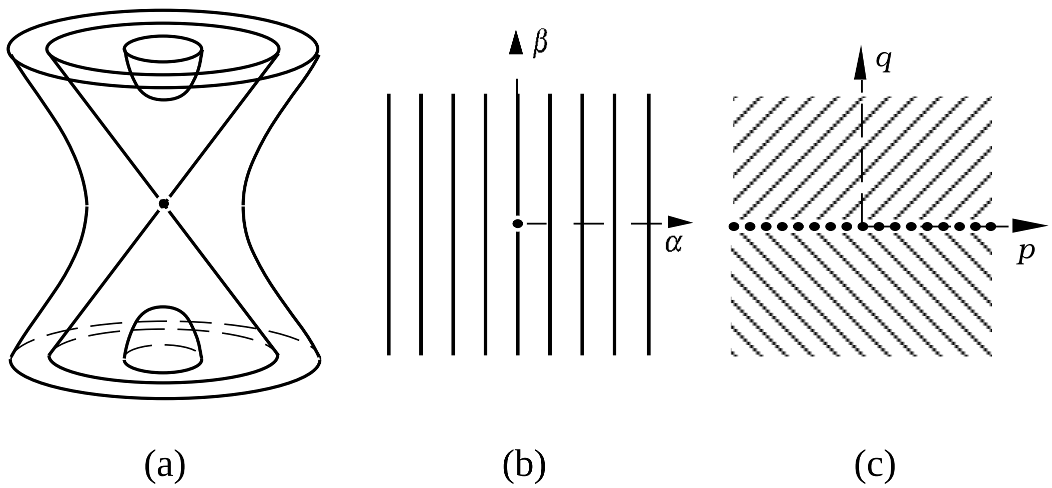

FIGURE 3. (a) The (co)adjoint orbits in the matrix algebra $\mathfrak{sl}(2, \mathbb{R})$ are the connected components of the quadrics. The adjoint (b) and coadjoint (c) orbits of the group of affine transformations of $\mathbb{R}$.

D) The adjoint orbits of the group $G = \{x \mapsto ax + b \mid a > 0,\ b \in \mathbb{R}\}$ of affine transformations of the real line $\mathbb{R}$ are straight lines $\{\alpha = \text{ const } \neq 0\}$, two rays $\{\alpha = 0,\ \beta > 0\}, \{\alpha = 0,\ \beta < 0\}$, and the origin $\{\alpha = 0,\ \beta = 0\}$ in the plane $\{(\alpha, \beta)\} = \mathfrak{g}$, Fig.3b .

E) Let v be a divergence-free vector field on M. The adjoint orbit of v for the group $S\text{Diff}(M)$ consists of the divergence-free vector fields obtained from v by the natural action of all diffeomorphisms preserving the volume element in the domain M. In particular, all such fields are topologically equivalent. For instance, they have equal numbers of stagnation points, of periodic orbits, of invariant surfaces, the same eigenvalues of linearizations at fixed points, etc.

REMARK 2.19. For a simply connected bounded domain M in the (x, y)-plane, a divergence-free vector field tangent to the boundary of M can be defined by its *stream function* ψ (such that the field components are $-\psi_y$ and ψ_x). One can assume that the stream function vanishes on the boundary of M. The Lie algebra of the group $S\text{Diff}(M)$, which consists of diffeomorphisms preserving the area element of the domain M, is naturally identified with the space of all such stream functions ψ.

THEOREM 2.20. *All momenta* $I_n = \iint_M \psi^n dxdy$ *are constant along the adjoint orbits of the group* $S\,\mathrm{Diff}(M)$ *in the space of stream functions.*

PROOF. Along every orbit all the areas $S(c)$ of the sets "of smaller values" $\{(x,y) \mid \psi(x,y) < c\}$ are constant. □

REMARK 2.21. Besides the above quantities, neither a topological type of the function ψ (in particular, the number of singular points, configuration of saddle separatrices, etc.) nor the areas bounded by connected components of level curves of the stream function ψ change along the orbits, see Fig.4.

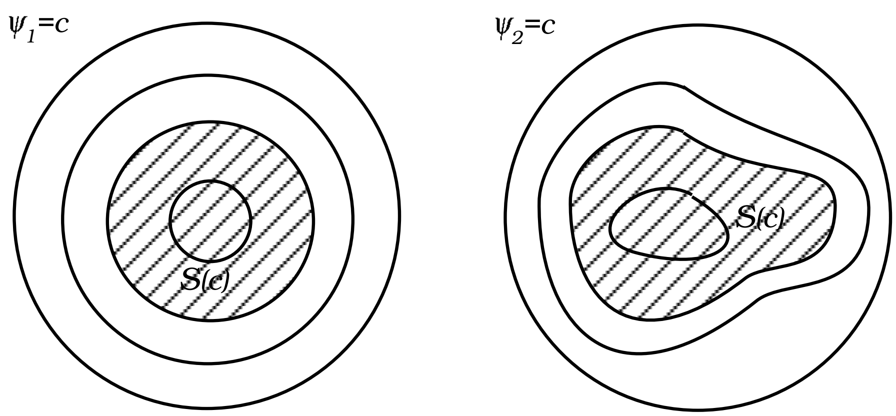

FIGURE 4. The stream functions of the fields from the same adjoint orbit have equal areas of "smaller values" sets.

The periods of particle motion along corresponding closed trajectories are constant under the diffeomorphism action as well. However, the latter invariant can be expressed in terms of the preceding ones. For instance, the period of motion along the closed trajectory $\psi = c$, which bounds a topological disk of area $S(c)$, is given by the formula $T = \frac{\partial S}{\partial c}$.

§3. Coadjoint representation of a Lie group

The main battlefield of the Eulerian hydrodynamics of an ideal fluid, as well as of the Eulerian dynamics of a rigid body, is not the Lie algebra, but the corresponding dual space, not the space of adjoint representation, but that of coadjoint representation of the corresponding group.

3.A. Definition of the coadjoint representation. Consider the vector space $\mathfrak{g}^*$ dual to a Lie algebra $\mathfrak{g}$. Vectors of $\mathfrak{g}^*$ are linear functions on the space of the Lie algebra $\mathfrak{g}$. The space $\mathfrak{g}^*$, in general, does not have a natural structure of a Lie algebra.

EXAMPLE 3.1. Every *component* of the vector of angular velocity of a rigid body is a *vector* of the space dual to the Lie algebra $\mathfrak{so}(3)$.

To every linear operator $A : X \to Y$ one can associate the dual (or adjoint) operator acting in the reverse direction, between the corresponding dual spaces, $A^* : Y^* \to X^*$, and defined by the formula

$$(A^* y)(x) = y(Ax)$$

for every $x \in X$, $y \in Y^*$. In particular, the differentials of the left and right translations

$$L_{g*} : T_h G \to T_{gh} G, \qquad R_{g*} : T_h G \to T_{hg} G$$

define the dual operators

$$L_g^* : T_{gh}^* G \to T_h^* G, \qquad R_g^* : T_{hg}^* G \to T_h^* G.$$

DEFINITION 3.2. The *coadjoint (anti)representation* of a Lie *group* G in the space $\mathfrak{g}^*$ dual to the Lie algebra $\mathfrak{g}$ is the (anti)representation that to each group element g associates the linear transformation

$$\mathrm{Ad}_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$$

dual to the transformation $Ad_g : \mathfrak{g} \to \mathfrak{g}$. In other words,

$$(\mathrm{Ad}_g^* \,\xi)(\omega) = \xi(\mathrm{Ad}_g \,\omega)$$

for every $g \in G, \xi \in \mathfrak{g}^*, \omega \in \mathfrak{g}$. The operators Ad_g^* form an *anti*representation, since $\mathrm{Ad}_{gh}^* = \mathrm{Ad}_h^* \,\mathrm{Ad}_g^*$.

The orbit of a point $\xi \in \mathfrak{g}^*$ under the action of the coadjoint representation of a group G (in short, the *coadjoint orbit* of ξ) is the set of all points $\mathrm{Ad}_g^* \,\xi$ $(g \in G)$ in the space $\mathfrak{g}^*$ dual to the Lie algebra $\mathfrak{g}$ of the group G.

For the group $SO(3)$ the coadjoint orbits are spheres centered at the origin of the space $\mathfrak{so}(3)^*$. They are similar to the adjoint orbits of this group, which are spheres in the space $\mathfrak{so}(3)$. However, in general, the coadjoint and adjoint representations are not alike.

EXAMPLE 3.3. Consider the group G of all affine transformations of a line $G = \{x \mapsto ax + b \mid a > 0, \ b \in \mathbb{R}\}$. The coadjoint representation acts on the plane $\mathfrak{g}^*$ of linear functions $p\,da + q\,db$ at the group unity $(a = 1, b = 0)$. The orbits of the coadjoint representation are the upper $(q > 0)$ and lower $(q < 0)$ half-planes, as well as every single point $(p, 0)$ of the axis $q = 0$ (see Fig.3c).

DEFINITION 3.4. The *coadjoint representation* of an element v of a Lie *algebra* $\mathfrak{g}$ is the rate of change of the operator $\mathrm{Ad}^*_{g_t}$ of the coadjoint group representation as the group element g_t leaves the unity $g_0 = e$ with velocity $\dot{g} = v$. The operator of the coadjoint representation of the algebra element $v \in \mathfrak{g}$ is denoted by

$$\mathrm{ad}^*_v : \mathfrak{g}^* \to \mathfrak{g}^*.$$

It is dual to the operator of the adjoint representation $\mathrm{ad}^*_v \omega(u) = \omega(\mathrm{ad}_v u) = \omega([v, u])$ for every $v \in \mathfrak{g}, u \in \mathfrak{g}, \omega \in \mathfrak{g}^*$. Given $\omega \in \mathfrak{g}^*$, the vectors $\mathrm{ad}^*_v \omega$, with various $v \in \mathfrak{g}$, constitute the tangent space to the coadjoint orbit of the point (similar to the fact that the vectors $\mathrm{ad}_v u, v \in \mathfrak{g}$ form the tangent space to the adjoint orbit of the point $u \in \mathfrak{g}$).

3.B. Dual of the space of plane divergence-free vector fields. Look at the group $G = S\mathrm{Diff}(M)$ of diffeomorphisms preserving the area element of a connected and simply connected bounded domain M in the $\{(x, y)\}$-plane. The corresponding Lie algebra is identified with the space of stream functions, i.e., of smooth functions in M vanishing on the boundary. The identification is natural in the sense that it does not depend on the Euclidean structure of the plane, but it relies solely on the area element μ on M.

DEFINITION 3.5. The *inner product* of a vector v with a differential k-form ω is the $(k-1)$-form $i_v\omega$ obtained by substituting the vector v into the form ω as the first argument:

$$(i_v\omega)(\xi_1, \ldots, \xi_{k-1}) = \omega(v, \xi_1, \ldots, \xi_{k-1}).$$

DEFINITION 3.6. The *vector field* v, with a *stream function* ψ on a surface with an area element μ, is the field obeying the condition

$$i_v\mu = -d\psi. \tag{3.1}$$

For instance, suppose that (x, y) are coordinates in which $\mu = dx \wedge dy$.

LEMMA 3.7. *The components of the field with a stream function ψ in the above coordinate system are*

$$v_x = -\frac{\partial \psi}{\partial y}, \quad v_y = \frac{\partial \psi}{\partial x}.$$

PROOF. For an arbitrary vector u, the following identity holds by virtue of the definition of $\mu = dx \wedge dy$:

$$(i_v\mu)(u) = \mu(v, u) = \begin{vmatrix} v_x & v_y \\ dx(u) & dy(u) \end{vmatrix} = (v_x\,dy - v_y\,dx)(u).$$

□

Condition (3.1) determines the stream function up to an additive constant. The latter is defined by the requirement $\psi\mid_{\partial M}=0$.

The space dual to the space of all divergence-free vector fields v can also be described by means of smooth functions on M, however, not necessarily vanishing on ∂M. Indeed, it is natural to interpret the objects dual to vector fields in M as differential 1-forms α on M. The value of the corresponding linear function on a vector field v is

$$\alpha \mid v := \iint_M \alpha(v)\mu.$$

One readily verifies the following

LEMMA 3.8. *1) If α is the differential of a function, then $\alpha \mid v = 0$ for every divergence-free field v on M tangent to ∂M.*

2) Conversely, if $\alpha \mid v = 0$ for every divergence-free field v on M tangent to ∂M, then α is the differential of a function on M.

3) If for a given v, one has $\alpha \mid v = 0$ for α the differential of every function on M, then the vector field v is divergence-free and tangent to the boundary ∂M.

The proof of this lemma in a more general situation of a (not necessarily simply connected) manifold of arbitrary dimension is given in Section 8. This lemma manifests the formal identification of the space $\mathfrak{g}^*$ dual to the Lie algebra of divergence-free vector fields in M tangent to the boundary ∂M with the quotient space $\Omega^1(M)/d\Omega^0(M)$ (of all 1-forms on M modulo full differentials). Below we use the identification in this "formal" sense. In order to make precise sense of the discussed duality according to the standards of functional analysis, one has to specify a topology in one of the spaces and to complete the other accordingly. Here we will not fix the completions, and we will regard the elements of both of $\mathfrak{g}$ and $\mathfrak{g}^*$ as smooth functions (fields, forms) unless otherwise specified.

LEMMA 3.9. *Let M be a two-dimensional simply connected domain with an area form μ. Then the map $\alpha \mapsto f$ given by*

$$d\alpha = f\mu$$

(where μ is the fixed area element and α is a 1-form in M) defines a natural isomorphism of the space $\Omega^1/d\Omega^0 = \mathfrak{g}^*$, dual to the Lie algebra $\mathfrak{g}$ of divergence-free fields in M (tangent to its boundary ∂M), and the space of functions f in M.

PROOF. Adding a full differential to α does not change the function f. Hence we have constructed a map of $\mathfrak{g}^* = \Omega^1/d\Omega^0$ to the space of functions f on M. Since

M is simply connected, every function f is the image of a certain closed 1-form α, determined modulo the differential of a function. □

THEOREM 3.10. *The coadjoint representation of the group* $S\,\mathrm{Diff}(M)$ *in* $\mathfrak{g}^*$ *is the natural action of diffeomorphisms, preserving the area element of* M*, on functions on* M.

PROOF. It follows from the fact that all our identifications are natural, i.e., invariant with respect to transformations belonging to $S\mathrm{Diff}(M)$. □

3.C. The Lie algebra of divergence-free vector fields and its dual in arbitrary dimension. Let $G = S\mathrm{Diff}(M^n)$ be the group of diffeomorphisms preserving a volume element μ on a manifold M with boundary ∂M (in general, M is of any dimension n and multiconnected, but it is assumed to be compact).

The commutator $[v, w]$ (or, $L_v w$) in the corresponding Lie algebra of divergence-free vector fields on M tangent to ∂M is given by minus their Poisson bracket: $[v, w] = -\{v, w\}$, see Example 2.9.

THEOREM 3.11 (SEE E.G., [M-W]). *The Lie algebra* $\mathfrak{g}$ *of the group* $G = S\,\mathrm{Diff}(M)$ *is naturally identified with the space of closed differential* $(n-1)$*-forms on* M *vanishing on* ∂M*. Namely, a divergence-free field* v *is associated to the* $(n-1)$*-form* $\omega_v = i_v\mu$*. The dual space* $\mathfrak{g}^*$ *to the Lie algebra* $\mathfrak{g}$ *is* $\Omega^1(M)/d\Omega^0(M)$*. The adjoint and coadjoint representations are the standard actions of the diffeomorphisms on the corresponding differential forms.*

The proof is given in Section 8.

EXAMPLE 3.12. Let M be a three-dimensional simply connected domain with boundary. Consider the group $S\mathrm{Diff}(M)$ of diffeomorphisms preserving the volume element μ (for simply connected M, this group coincides with the group of so-called *exact diffeomorphisms*; see Section 8). Its Lie algebra $\mathfrak{g}$ consists of divergence-free vector fields in M tangent to the boundary ∂M. In the simply connected case the dual space $\mathfrak{g}^* = \Omega^1(M)/d\Omega^0(M)$ can be identified with *all closed 2-forms* in M by taking the differential of the forms from $\Omega^1(M)$.

We will see below that the vorticity field for a flow with velocity $v \in \mathfrak{g}$ in M is to be regarded as an element of the dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$. The reason is that every 2-form that is the differential of a 1-form corresponds to a certain vorticity field.

On a non-simply connected manifold, the space $\mathfrak{g}^*$ is somewhat bigger than the set of vorticities. In the latter case the physical meaning of the space $\mathfrak{g}^*$, dual to the Lie algebra $\mathfrak{g}$, is the space of circulations over all closed curves. The vorticity field determines the circulations of the initial velocity field over all curves that *are boundaries* of two-dimensional surfaces lying in the domain of the flow. Besides the above, a vector from $\mathfrak{g}^*$ keeps the information about circulation over all other closed curves that *are not boundaries* of anything.

§4. Left-invariant metrics and a rigid body for an arbitrary group

A Riemannian metric on a Lie group G is *left-invariant* if it is preserved under every left shift L_g. The left-invariant metric is defined uniquely by its restriction to the tangent space to the group at the unity, i.e., by a quadratic form on the Lie algebra $\mathfrak{g}$ of the group.

Let $A : \mathfrak{g} \to \mathfrak{g}^*$ be a symmetric positive definite operator that defines the inner product

$$\langle \xi, \eta \rangle = (A\xi, \eta) = (A\eta, \xi)$$

for any ξ, η in $\mathfrak{g}$. (Here the round brackets stand for the pairing of elements of the dual spaces $\mathfrak{g}$ and $\mathfrak{g}^*$.) The positive-definiteness of the quadratic form is not very essential, but in many applications, such as motion of a rigid body or hydrodynamics, the corresponding quadratic form plays the role of kinetic energy.

Definition 4.1. The operator A is called the *inertia operator.*

Define the symmetric linear operator $A_g : T_gG \to T_g^*G$ at every point g of the group G by means of the left translations from g to the unity:

$$A_g\xi = L_g^{*-1} A L_{g^{-1}*}\xi.$$

At every point g, we obtain the inner product

$$\langle \xi, \eta \rangle_g = (A_g\xi, \eta) = (A_g\eta, \xi) = \langle \eta, \xi \rangle_g,$$

where $\xi, \eta \in T_gG$. This product determines the left-invariant Riemannian metric on G. Thus we obtain the commutative diagram in Fig.5.

Example 4.2. For a classical rigid body with a fixed point, the configuration space is the group $G = SO(3)$ of rotations of three-dimensional Euclidean space. A motion of the body is described by a curve $t \mapsto g(t)$ in the group. The Lie algebra $\mathfrak{g}$ of the group G is the three-dimensional space of angular velocities of all possible rotations. The commutator in this Lie algebra is the usual cross product.

A rotation velocity $\dot{g}(t)$ of the body is a tangent vector to the group at the point $g(t)$. By translating it to the identity via left or right shifts, we obtain two elements of the Lie algebra $\mathfrak{g}$.

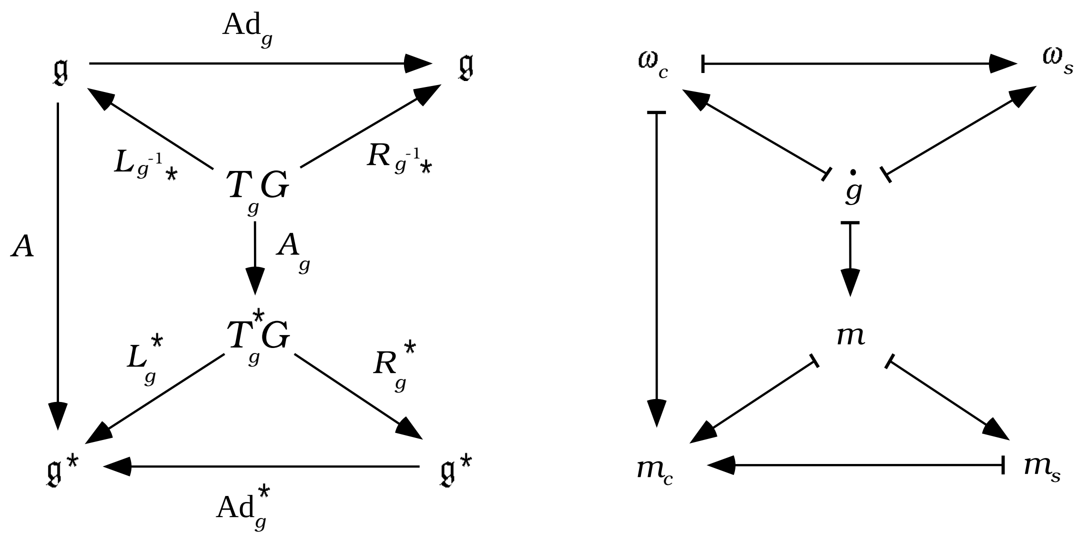

FIGURE 5. Diagram of the operators in a Lie algebra and in its dual.

DEFINITION 4.3. The result of the left translation is called *the angular velocity in the body* (and is denoted by ω_c with c for "corps" = body), while the result of the right translation is the *spatial angular velocity* (denoted by ω_s),

$$\omega_c = L_{g^{-1}*}\dot{g} \in \mathfrak{g}, \qquad \omega_s = R_{g^{-1}*}\dot{g} \in \mathfrak{g}.$$

Note that $\omega_s = \mathrm{Ad}_g\, \omega_c$.

The space $\mathfrak{g}^*$, dual to the Lie algebra $\mathfrak{g}$, is called the *space of angular momenta.* The symmetric operator $A : \mathfrak{g} \to \mathfrak{g}^*$ is the *operator* (or tensor) *of inertia momentum.* It is related to the kinetic energy E by the formula

$$E = \frac{1}{2}\langle \dot{g}, \dot{g}\rangle_g = \frac{1}{2}\langle \omega_c, \omega_c\rangle = \frac{1}{2}(A\omega_c, \omega_c) = \frac{1}{2}(A_g\dot{g}, \dot{g}).$$

The image m of the velocity vector $\dot{g}$ under the action of the operator A_g belongs to the space T^*_gG. This vector can be carried to the cotangent space to the group G at the identity by both left or right translations. The vectors

$$m_c = L^*_g m \in \mathfrak{g}^*, \qquad m_s = R^*_g m \in \mathfrak{g}^*$$

are called the vector of the *angular momentum relative to the body* (m_c) and that of the *angular momentum relative to the space* (or *spatial angular momentum*, m_s). Note that $m_c = \mathrm{Ad}^*_g\, m_s$.

The kinetic energy is given by the formula

$$E = \frac{1}{2}(m_c, \omega_c) = \frac{1}{2}(m, \dot{g})$$

in terms of momentum and angular velocity. The quadratic form E defines a left invariant Riemannian metric on the group. According to the least action principle, inertia motions of a rigid body with a fixed point are geodesics on the group $G = SO(3)$ equipped with this left-invariant Riemannian metric. (Note that in the case $SO(3)$ of the motion in three-dimensional space, the inertia operators of genuine rigid bodies form an open set in the space of all symmetric operators $A : \mathfrak{g} \to \mathfrak{g}^*$ (some triangle inequality should be satisfied).)

Similarly, in the general situation of a left-invariant metric on an arbitrary Lie group G, we consider four vectors moving in the spaces $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively:

$$\omega_c(t) \in \mathfrak{g}, \quad \omega_s(t) \in \mathfrak{g}, \quad m_c(t) \in \mathfrak{g}^*, \quad m_s(t) \in \mathfrak{g}^*.$$

They are called the vectors of angular velocity and momentum in the body and in space.

L. Euler [Eul] found the differential equations that these moving vectors satisfy:

Theorem 4.4 (First Euler Theorem). *The vector of spatial angular momentum is preserved under motion:*

$$\frac{d\,m_s}{dt} = 0.$$

Theorem 4.5 (Second Euler Theorem). *The vector of angular momentum relative to the body obeys the Euler equation*

$$\frac{d\,m_c}{dt} = \mathrm{ad}^*_{\omega_c} m_c. \tag{4.1}$$

Remark 4.6. The vector $\omega_c = A^{-1} m_c$ is linearly expressed in terms of m_c. Therefore, the Euler equation defines a quadratic vector field in $\mathfrak{g}^*$, and its flow describes the evolution of the vector m_c. The latter evolution of the momentum vector depends only on the position of the momentum vector in the body, but not in the ambient space.

In other words, the geodesic flow in the phase manifold T^*G is fibered over the flow of the Euler equation in the space $\mathfrak{g}^*$, whose dimension is one half that of T^*G.

Proofs. Euler proved his theorems for the case of $G = SO(3)$, but the proofs are almost literally applicable to the general case. Namely, the First Euler Theorem is the conservation law implied by the energy symmetry with respect to left translations. The Second Euler Theorem is a formal corollary of the first and of the identity

$$m_c(t) = \mathrm{Ad}^*_{g(t)} m_s. \tag{4.2}$$

Differentiating the left- and right-hand sides of the identity in t at $t = 0$ (and assuming that $g(0) = e$), we obtain the Euler equation (4.1) for this case. The left-invariance of the metric implies that the right-hand side depends solely on m_c, but not on $g(t)$, and therefore the equation is satisfied for every $g(t)$. □

REMARK 4.7. The Euler equation (4.1) for a rigid body in $\mathbb{R}^3$ is $\dot{m} = m \times \omega$ for the angular momentum $m = A\omega$. For $A = \operatorname{diag}(I_1, I_2, I_3)$ one has

$$\begin{cases} \dot{m}_1 = \gamma_{23} m_2 m_3, \\ \dot{m}_2 = \gamma_{31} m_3 m_1, \\ \dot{m}_3 = \gamma_{12} m_1 m_2, \end{cases}$$

where $\gamma_{ij} = I_j^{-1} - I_i^{-1}$. The principal inertia momenta I_i satisfy the triangle inequality $|I_i - I_j| \le I_k$.

The relation (4.2) and the First Euler Theorem imply the following

THEOREM 4.8. *Each solution $m_c(t)$ of the Euler equation belongs to the same coadjoint orbit for all t. In other words, the group coadjoint orbits are invariant submanifolds for the flow of the Euler equation in the dual space $\mathfrak{g}^*$ to the Lie algebra.*

The isomorphism $A^{-1} : \mathfrak{g}^* \to \mathfrak{g}$ allows one to rewrite the Euler equation on the Lie algebra as an evolution law on the vector $\omega_c = A^{-1} m_c$. The result is as follows.

THEOREM 4.9. *The vector of angular velocity in the body obeys the following equation with quadratic right-hand side:*

$$\frac{d\omega_c}{dt} = B(\omega_c, \omega_c),$$

where the bilinear (nonsymmetric) form $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is defined by

$$\langle [a, b], c \rangle = \langle B(c, a), b \rangle \tag{4.3}$$

for every a, b, c in $\mathfrak{g}$. Here, $[\cdot,\cdot]$ is the commutator in the Lie algebra $\mathfrak{g}$, and $\langle \cdot,\cdot \rangle$ is the inner product in the space $\mathfrak{g}$.

REMARK 4.10. The operation B is bilinear, and for a fixed first argument, it is skew symmetric with respect to the second argument:

$$\langle B(c, a), b \rangle + \langle B(c, b), a \rangle = 0.$$

The operator B is the image of the operator of the algebra coadjoint representation under the isomorphism of $\mathfrak{g}$ and $\mathfrak{g}^*$ defined by the inertia operator A.

PROOF OF THEOREM 4.9. For each $b \in \mathfrak{g}$, we have

$$\langle \frac{d\omega_c}{dt}, \ b\rangle = \langle A^{-1}\frac{d\,m_c}{dt}, \ b\rangle = \frac{d\,m_c}{dt} \mid b,$$

where $\langle\cdot,\cdot\rangle$ is the inner product in the Lie algebra, and $.|.$ stands for the pairing of elements from $\mathfrak{g}$ and $\mathfrak{g}^*$. By virtue of the Euler equation,

$$\frac{d\,m_c}{dt} \mid b = (\mathrm{ad}^*_{\omega_c}\, m_c) \mid b = m_c \mid \mathrm{ad}_{\omega_c}\, b = m_c \mid [\omega_c, b].$$

By definition of the inner product,

$$m_c \mid [\omega_c, b] = (A\omega_c) \mid [\omega_c, b] = \langle [\omega_c, b], \omega_c\rangle.$$

The definition of the operation B allows one to rewrite it as

$$\langle [\omega_c, b], \omega_c\rangle = \langle B(\omega_c, \omega_c), b\rangle.$$

Thus, for each b we finally have

$$\langle \frac{d\omega_c}{dt}, b\rangle = \langle B(\omega_c, \omega_c), b\rangle,$$

which proves Theorem 4.9. □

REMARK 4.11. Consider the motion of a three-dimensional rigid body. The Euler equation (4.1) describes the evolution of the momentum vector in the three-dimensional space $\mathfrak{so}(3,\mathbb{R})^*$. Each solution $m_c(t)$ of the Euler equation belongs to the intersection of the coadjoint orbits (which are spheres centered at the origin) with the the energy levels, see Fig.6. The kinetic energy is a quadratic first integral on the dual space, and its level surfaces are ellipsoids $\langle A^{-1}m_c, m_c\rangle = \mathrm{const}$.

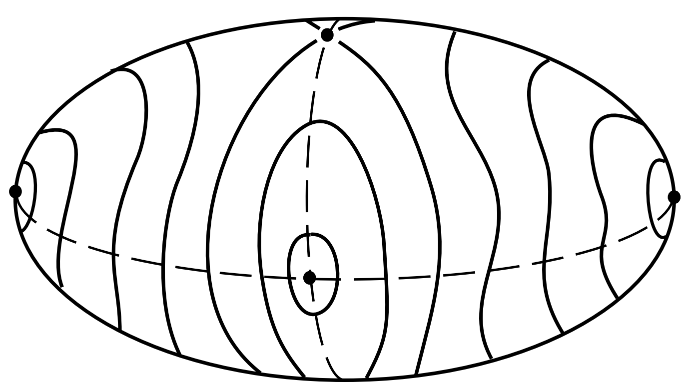

FIGURE 6. Trajectories of the Euler equation on an energy level surface.

The dynamics of an n-dimensional rigid body is naturally associated to the group $SO(n,\mathbb{R})$. The trajectories of the corresponding Euler equation are no longer determined by the intersections of the coadjoint orbits of this group with the energy levels (see Section VI.1.B).

In the next Section we will apply the Euler Theorems to the (infinite-dimensional) group of volume-preserving diffeomorphisms [Arn4,16]. Note that the analogy between the Euler equations for ideal hydrodynamics and for a rigid body was pointed out by Moreau in [Mor1].

§5. Applications to hydrodynamics

According to the principle of least action, motions of an ideal (incompressible, inviscid) fluid in a Riemannian manifold M are geodesics of a *right*-invariant metric on the Lie group $S\mathrm{Diff}(M)$. Such a metric is defined by the quadratic form E (E being the kinetic energy) on the Lie algebra of divergence-free vector fields:

$$E = \frac{1}{2}\iint\limits_M v^2\ \mu,$$

where μ is a volume element on M, and v^2 is the square of Riemannian length of a vector tangent to M.

REMARK 5.1. To carry out the passage from left-invariant metrics to right-invariant ones, it suffices to change the sign of the commutator $[\ ,\ ]$ (as well as of all operators linearly depending on it: $\mathrm{ad}_v\cdot = [v,\cdot]$, ad_v^*, B) in all the formulas. Indeed, the Lie group G remains a group after the change of the product $(g,h)\mapsto gh$ to $(g,h)\mapsto g*h = hg$.

The Lie algebra commutator changes sign under this transform, while a left-invariant metric becomes right invariant. Of course, left translations with respect to the old group operation become right translations for the new one. Therefore, for right-invariant metrics the result of the *right* translation of a momentum vector to the dual Lie algebra is preserved in time, while the left translation of the momentum obeys the Euler equation.

In hydrodynamics the metric on the group is right invariant. Hence, from the general results of the preceding section we obtain the (Euler) equations of motion of an ideal fluid (on a Riemannian manifold of arbitrary dimension), as well as the conservation laws for them.

The Euler equations on a flow *velocity field* in the domain M are the result of a *right* shift to the *Lie algebra* $\mathfrak{g} = S\mathrm{Vect}(M)$ of divergence-free vector fields on

M (see Theorem 4.8, with the change of the left shift to the right one). The right invariance of the metric results in the following form of the Euler equation:

$$\dot{v} = -B(v, v),$$

where the operation B on the Lie algebra $\mathfrak{g}$ is defined by (4.3). Its equivalent form is the *Euler–Helmholtz equation* on the *vorticity field*, i.e., Equation (4.1) with the opposite sign for right shifts of *momentum* to the *dual space* $\mathfrak{g}^*$ of the Lie algebra.

EXAMPLE 5.2. Consider the Lie algebra $\mathfrak{g} = S\mathrm{Vect}(M)$ of divergence-free vector fields on a simply connected domain M, tangent to ∂M, with the commutator $[\cdot,\cdot] = -\{\cdot,\cdot\}$ being minus the Poisson bracket. Below we show that the operation B for the Euler equation on this Lie algebra has the form

$$B(c, a) = \text{curl } c \times a + \text{ grad } p, \tag{5.1}$$

where $\times$ is the cross product and p is a function on M, determined uniquely (modulo an additive constant) by the condition $B \in \mathfrak{g}$ (i.e., by the conditions div $B = 0$ and tangency of $B(c, a)$ to ∂M). Hence, the *Euler equation for three-dimensional ideal hydrodynamics* is the evolution

$$\frac{\partial v}{\partial t} = v \times \text{curl } v - \text{ grad } p \tag{5.2}$$

of a divergence-free vector field v in $M \subset \mathbb{R}^3$ tangent to ∂M.

The *vortex* (or the *Euler–Helmholtz*) *equation* is as follows:

$$\frac{\partial \omega}{\partial t} = -\{v, \omega\}, \qquad \omega = \text{ curl } v. \tag{5.3}$$

PROOF. By definition of the operation B,

$$\langle B(c, a), b\rangle = \langle [a, b], c\rangle,$$

where $[a, b]$ is the commutator in the Lie algebra $S\mathrm{Vect}(M)$ (equal to $-\{a, b\}$ in terms of the Poisson bracket). Since all fields are divergence free, we have

$$\langle [a, b], c\rangle = \langle \text{curl } (a \times b), c\rangle = \langle a \times b, \text{curl } c\rangle = \langle (\text{curl } c) \times a, b\rangle.$$

Thus, curl $c \times a$ gives the explicit form of the operation B, modulo a gradient term (since div $b = 0$).

The vortex equation is obtained from the Euler equation on the velocity field by taking curl of both sides. □

Formula (5.1) holds in a more general situation of a Riemannian three-dimensional manifold M with boundary. Moreover, for a manifold of arbitrary dimension, one can still make sense of this formula by specifying the definition of the cross product.

THEOREM 5.3. *The operation $B(v, v)$ for a divergence-free vector field v on a Riemannian manifold M of any dimension is*

$$B(v, v) = \nabla_v v + \text{ grad } p.$$

Here $\nabla_v v$ is the vector field on M that is the covariant derivative of the field v along itself in the Riemannian connection on M related to the chosen Riemannian metric, and p is determined modulo a constant by the same conditions as above.

We postpone the proof of this theorem until the discussion of covariant derivative in Section IV.1. The proof is based on the following simple interpretation of the inertia operator for hydrodynamics. As we discussed above, the Lie algebra of divergence-free vector fields and its dual space can be defined as soon as the manifold is equipped with a volume form. The inertia operator requires an additional structure, a *Riemannian metric on the manifold*, similar to fixing an inertia ellipsoid for a rigid body.

THEOREM 5.4. *The inertia operator for ideal hydrodynamics on a Riemannian manifold takes a velocity vector field to the 1-form whose value on an arbitrary vector equals the Riemannian inner product of the latter vector with the velocity vector at that point (the obtained 1-form is regarded modulo the differentials of functions).*

See the proof in Section 7 (Theorem 7.19).

In the case of hydrodynamics, the invariance of coadjoint orbits with respect to the Euler dynamics (Theorem 4.8) takes the form of Helmholtz's classical theorem on vorticity conservation.

THEOREM 5.5. *The circulation of any velocity field over each closed curve is equal to the circulation of this velocity field, as it changes according to the Euler equation, over the curve transported by the fluid flow.*

PROOF. Consider an element of the Lie algebra $S\text{Vect}(M)$ corresponding to a "narrow current" that flows along the chosen curve and has unit flux across a transverse to the curve. Under the adjoint representation (i.e., action of a volume-preserving diffeomorphism), this element is taken to a similar "narrow current" along the transported curve.

The pairing of a vector of the dual Lie algebra with the chosen element in the Lie algebra itself is the integral of the corresponding 1-form along the curve (note that although an element of the dual space is a 1-form modulo any function differential, its integral over a closed curve is well-defined). By Theorem 5.4, the latter pairing is the circulation of the velocity field along our curve. □

The above theorem implies that the velocity fields (parametrized by time t) that constitute one solution of the Euler equation are isovorticed; i.e., the vorticity of the field at any given moment of time t is transported to the vorticity at any other moment by a diffeomorphism preserving the volume element.

REMARK 5.6. Isovorticity, i.e., the condition on phase points to belong to the same coadjoint orbit, imposes constraints that differ drastically in two- and three-dimensional cases. For a two-dimensional fluid the coadjoint orbits are distinguished by the values of the first integrals, such as vorticity momenta. In the three-dimensional case the orbit geometry is much more subtle.

Owing to this difference in the geometry of coadjoint orbits, the foundation of three-dimensional hydrodynamics encounters serious difficulties. Meanwhile, in the hydrodynamics of a two-dimensional fluid, the existence and uniqueness of global solutions have been proved [Yu1], and the proofs use heavily the first integrals of the Euler equation, which are invariant on the coadjoint orbits.

DEFINITION 5.7. Given a velocity vector field, consider the 1-form that is the (pointwise) Riemannian inner product with the velocity field. Its differential is called the *vorticity form.*

EXAMPLE 5.8. On the Euclidean plane (x, y) this 2-form is $\omega\, dx \wedge dy$, where ω is a function. The function ω, also called the vorticity of a two-dimensional flow, is related to the stream function ψ by the identity $\omega = \Delta\psi$.

In three-dimensional Euclidean space this is the 2-form corresponding to the vorticity vector field curl v. Its value on a pair of vectors equals their mixed product with curl v.

DEFINITION 5.9. The *vorticity vector field* of an incompressible flow on a three-dimensional Riemannian manifold is defined as the vector field ξ associated to the vorticity 2-form ω according to the formula

$$\omega = i_\xi \mu,$$

where μ is the volume element. In other words, the vorticity vector ξ is defined at each point by the condition

$$\mu(\xi, a, b) = \omega(a, b) \tag{5.4}$$

for any pair of vectors a, b attached at that point. One has to note that the construction of the field ξ does not use any coordinates or metric but only the volume element μ and the 2-form ω.

REMARK 5.10. On a manifold of an arbitrary dimension n the vorticity is not a vector field but an $(n-2)$-polyvector field (*k-polyvector*, or k-vector, is a polylinear skew-symmetric function of k cotangent vectors, i.e., of k 1-forms at the point). For instance, for $n = 2$, one obtains the 0-polyvector, that is, a scalar. Such a scalar is the vorticity function ω of a two-dimensional flow in the example above. From Theorem 5.5 follows

COROLLARY 5.11. *The vorticity field is frozen into the incompressible fluid.*

Indeed, by virtue of Theorem 5.5, the vorticity 2-form ω is transported by the flow, since it is the differential of the 1-form "inner product with v," which is transported. The volume 3-form μ is also transported by the flow (since the fluid is incompressible).

In turn, the vorticity field ξ is defined by the forms ω and μ in an invariant way (without the use of a Riemannian metric) by formula (5.4). Therefore, this field is "frozen"; i.e., it is transported by fluid particles just as if the field arrows were drawn on the particles themselves: A stretching of a particle in any direction implies the stretching of the field in the same direction.

REMARK 5.12. Consider any diffeomorphism preserving the volume element (but *a priori* not related to any fluid flow). If such a diffeomorphism takes a vorticity 2-form ω_1 into a vorticity 2-form ω_2, then it transports the vorticity field ξ_1 corresponding to the first form to the vorticity field ξ_2 corresponding to the second.

If, however, one starts with a velocity field and then associates to it the corresponding vorticity field, the vorticity transported by an arbitrary diffeomorphism is not, in general, the vorticity for the velocity field obtained from the initial velocity by the diffeomorphism action. Theorem 5.5 states that the coincidence holds for the family of diffeomorphisms that is the Euler flow of an incompressible fluid with a given initial velocity field. In other words, the momentary velocity fields in the same Euler flow are isovorticed.

COROLLARY 5.13. *The vorticity trajectories are transported by an Eulerian fluid motion on a three-dimensional Riemannian manifold.*

In particular, every "vorticity tube" (i.e., a pencil of vorticity lines) is carried along by the flow. The Helmholtz theorem is closely related to this geometric corollary but is somewhat stronger (especially in the non-simply connected case).

REMARK 5.14. In the two-dimensional case, the isovorticity of velocity vector fields means that the vorticity function ω is transported by the fluid flow: A point where the vorticity $\Delta\psi$ was equal to ω at the initial moment is taken to a point with the same vorticity value at any other moment of time. In particular, all vorticity momenta

$$I_k = \iint (\Delta\psi)^k \; dx\, dy$$

are preserved, and so are the areas of the sets of smaller vorticity values:

$$S(c) = \iint\limits_{\Delta\psi \leq c} dx\, dy$$

(see, e.g., [Ob]). The same holds in a non-simply connected situation.

The conservation laws provided by the Helmholtz theorem are a bit stronger than the conservation of all the momenta, even if the two-dimensional manifold is simply connected. Namely, one claims that the whole "tree" of the vorticity function $\omega = \Delta\psi$ (that is, the space of components of the level sets; see Fig.7) is preserved, as well as the vorticity function ω, along with the measure on this tree. (The latter measure associates to every segment on the tree the total area of the corresponding vorticity levels.) Apparently, it is the complete set of invariants of typical coadjoint orbits (say, for $S\mathrm{Diff}(S^2)$).

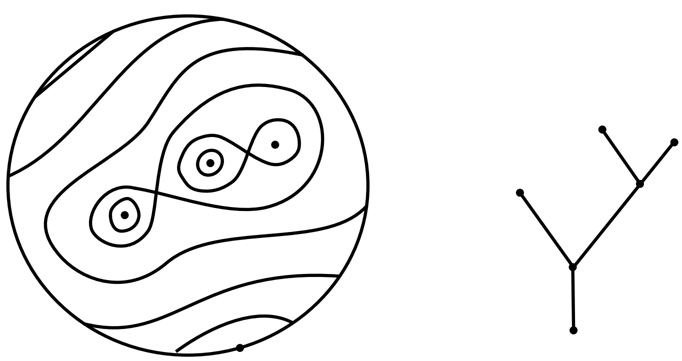

FIGURE 7. The "tree" of a vorticity function on a sphere.

It would be interesting to describe possible graphs for functions on non-simply connected manifolds, for instance, on a torus.

Applications of the Euler theorems to the hydrodynamics on manifolds of higher dimension are described in Section 7.

REMARK 5.15[1]. Though, as mentioned at the beginning, we are not dealing here with the existence and uniqueness theorems for the Euler equation of an ideal incompressible fluid, it is a very subtle question that has attracted considerable interest

[1] We are grateful to V. Yudovich for consulting us on the history of this question.

in the literature (see, e.g., [Chm, Gé, Yu3] for a survey). Local in time existence and uniqueness theorems of the classical solution of the basic initial boundary value problem for the two- and three-dimensional Euler equation were obtained in a series of papers by N. Gunter and L. Lichtenstein. W. Wolibner proved the global solvability for the 2D problem for the classical solutions (see [Ka] for the modern form of the result and generalizations). The global existence theorem in two-dimensional Eulerian hydrodynamics was proved by V. Yudovich [Yu1] for flows with vorticity in the space L^p for any given $p > 1$. For the uniqueness theorem on flows with essentially bounded vorticity and its generalizations; see [Yu1,3]; the nonuniqueness of weak solutions of the Euler equations is discussed in [Shn7].

If instead of an ideal fluid we consider a viscous incompressible one, its motion is described by the Navier–Stokes equation, being the Euler equation with an additional diffusion term; see Section 12. The local in time results (existence and uniqueness) for classical solutions of the Navier–Stokes equation were obtained by Lichtenstein, Odqvist, and Oseen (see the references in [Lad]). The global (in time and in "everything," i.e., the domain, initial field, and viscosity) existence of generalized solutions was proved by J. Leray (in the 1930s) and E. Hopf (in 1950/51). Uniqueness for this wide class of solutions is still unknown. The global existence and uniqueness theorems of generalized and classical solutions of the 2D Navier–Stokes equation were proved by Ladyzhenskaya and her successors (see [Lad]).

§6. Hamiltonian structure for the Euler equations

Recall the coadjoint representation of an arbitrary Lie group G. It turns out that the coadjoint orbits are always even-dimensional. The reason is that such an orbit is endowed with a natural symplectic structure (i.e., a closed nondegenerate 2-form). This structure, called the Kirillov (Berezin, Kostant) form (see [Ki1, Ber, Kos]), was essentially discovered by S. Lie [Lie].

The Euler equations in the dual space to a Lie algebra are Hamiltonian equations on each coadjoint orbit [Arn5]. Now the kinetic energy plays the role of the corresponding Hamiltonian function. We start with the following brief reminder.

Let (M, ω) be a symplectic manifold, i.e., a manifold M equipped with a closed nondegenerate differential 2-form. Recall that a Hamiltonian function H defines a Hamiltonian field v on M by the condition

$$i_v \omega = -dH. \tag{6.1}$$

In other words, the field v is the skew gradient of the function $H : M \to \mathbb{R}$ defined by the relation $-dH(\xi) = \omega(v, \xi)$ for every ξ (the ordinary gradient of a function is defined by the condition $dH(\xi) = \langle \text{grad } H, \xi\rangle$ for every ξ, where $\langle \cdot, \cdot\rangle$ is an inner product on a Riemannian manifold M). The value of the skew-symmetric 2-form ω on a pair of vectors (v and ξ in the case at hand) is called their skew-symmetric product. The following theorem is well known (see, e.g., [Arn16]).

THEOREM 6.1. *The phase flow of the Hamiltonian field v preserves the symplectic form ω and the Hamiltonian function H.*

Now assume that M is a coadjoint orbit of a Lie group G. The manifold M is embedded into the dual space $\mathfrak{g}^*$ of the corresponding Lie algebra. The tangent space to the orbit M at every point is spanned by the velocity vectors of the coadjoint representation corresponding to arbitrary velocities with which an element of the group G leaves the unity. In our notation (see Section 4), these vectors attached at a point $m \in M$ and tangent to M have the form

$$\xi = \mathrm{ad}_a^* m, \qquad m \in \mathfrak{g}^*, a \in \mathfrak{g}.$$

Now consider two such vectors, corresponding to two "angular velocities" (i.e., elements of the Lie algebra $\mathfrak{g}$)

$$\xi = \mathrm{ad}_a^* m,\ \eta = \mathrm{ad}_b^* m, \qquad a, b \in \mathfrak{g}.$$

One can combine these two vectors and one element of the dual space to get the number

$$\omega(\xi, \eta) := (m, [a, b]), \tag{6.2}$$

where the square brackets denote the commutator in the Lie algebra, and the round ones stand for the natural pairing between the dual spaces $\mathfrak{g}^*$ and $\mathfrak{g}$. One easily proves the following result.

THEOREM 6.2. *The value $\omega(\xi, \eta)$ depends on the vectors ξ and η tangent to M at m, but not on a particular choice of the "angular velocities" a and b used in the definition. The skew-symmetric form ω on M is closed and nondegenerate.*

This form defines the symplectic structure on the coadjoint orbit. It is invariant under the coadjoint representation (which follows from its definition).

EXAMPLE 6.3. For $G = SO(3)$ the coadjoint orbits are all spheres centered at the origin and the origin itself (note that all the dimensions are even!). Symplectic structures are the area elements invariant with respect to rotations of the spheres. The areas are normalized by the following condition: $\iint \omega$ is proportional to the sphere radius.

Formula (6.2) defines the symplectic structures on all coadjoint orbits at once. These symplectic structures are related in such a way that they equip the entire dual Lie algebra with a more general structure called the Poisson structure.

DEFINITION 6.4. A *Poisson structure* on a manifold is an operation $\{\cdot,\cdot\}$ that associates to a pair of smooth functions on the manifold a third one (their *Poisson bracket*) such that the operation is bilinear and skew-symmetric, and it satisfies the Jacobi identity

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$$

and the Leibniz identity

$$\{f, gh\} = \{f,g\}h + g\{f,h\}.$$

A manifold equipped with a Poisson structure is called a *Poisson manifold.*

The Leibniz identity means that for a fixed first argument, the operation $\{\cdot,\cdot\}$ is the differentiation of the second argument along some vector field.

EXAMPLE 6.5. Consider all smooth functions on the dual space $\mathfrak{g}^*$ of a finite-dimensional Lie algebra. Define the Poisson bracket on this space by

$$\{f,g\}(m) := (m, [df, dg]) \qquad \text{for } m \in \mathfrak{g}^*, f, g \in C^\infty(\mathfrak{g}^*), \tag{6.3}$$

where the differentials df and dg are taken at the same point m. Note that the differential of f at each point $m \in \mathfrak{g}^*$ is an element of the Lie algebra $\mathfrak{g}$ itself. Hence, the commutator $[df, dg]$ at every point is also a vector of this Lie algebra. The value of the linear function m evaluated at the latter vector, appearing on the right-hand side of the above formula, is, generally speaking, a nonlinear function of m, the Poisson bracket of the pair of functions f and g.

Let $x_1, \dots, x_n$ be coordinates in the dual space to an n-dimensional Lie algebra. Then formula (6.3) assumes the form

$$\{f,g\} = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}[x_i, x_j].$$

Here the vectors x_i form a basis of the Lie algebra $\mathfrak{g}$ itself. Their commutators lie in the Lie algebra as well, and therefore they are (linear) functions on the dual space $\mathfrak{g}^*$.

DEFINITION 6.6. The operation defined above is called the *natural Lie–Poisson structure* on the dual space to a Lie algebra.

One readily verifies (see, e.g., [We]) the following

THEOREM 6.7. *This is indeed a Poisson structure.*

REMARK 6.8. In fact, Poisson structures, in a somewhat more general situation, were introduced by Jacobi in "Lectures on dynamics" [Jac] while analyzing the structure of the ring of first integrals for a given Hamiltonian vector field. The Jacobi theory is more algebraic than topological, and it defines the Poisson structures on more general sets, similar to varieties of algebraic geometry rather than on the manifolds of topologists. Generally speaking, those sets are not Hausdorff. The modern definition was introduced by A. Weinstein [We], after the works of Lichnerowicz and Kirillov.

DEFINITION 6.9. The *Hamiltonian field* of a function H on a manifold equipped with a Poisson structure is the vector field ξ defined by the relation

$$L_\xi f = \{H, f\}$$

for every function f. Here $L_\xi f$ is the derivative of a function f along the vector field ξ_i, in coordinates $L_\xi f = \sum \xi_i \frac{\partial f}{\partial x_i}$.

EXAMPLE 6.10. The Hamiltonian field ξ of a linear function a on the dual space of a Lie algebra is given by the formula

$$\xi(\cdot) = \mathrm{ad}_a^* \cdot\, ,$$

where a is understood as a vector of the Lie algebra itself.

Indeed, at every point $m \in \mathfrak{g}^*$ and for every function f on $\mathfrak{g}^*$, one has

$$\{a, f\}(m) = (m, [a, df]) = (\mathrm{ad}_a^* m, df) = (L_\xi f)(m).$$

More generally, the Hamiltonian field ξ_H of an arbitrary smooth function H on the dual space g^* is given at a point $m \in g^*$ by the vector

$$\xi_H(m) = \mathrm{ad}_{dH}^* m,$$

where the differential dH is taken at the point m and is regarded as a vector of the Lie algebra.

REMARK 6.11. The Hamiltonian field $\xi_{\{F,H\}}$ associated to the Poisson bracket of two functions F and H is the Poisson bracket of the Hamiltonian fields ξ_F and ξ_H of these functions:

$$\{\xi_F, \xi_H\} = \xi_{\{F,H\}}.$$

It follows from definitions and the Jacobi identity that

$$\begin{aligned} L_{\{\xi_F,\xi_H\}} f &= L_{\xi_F} L_{\xi_H} f - L_{\xi_H} L_{\xi_F} f = L_{\xi_F}\{H, f\} - L_{\xi_H}\{F, f\} \\ &= \{F, \{H, f\}\} - \{H, \{F, f\}\} = -\{f, \{F, H\}\} = L_{\xi_{\{F,H\}}} f. \end{aligned}$$

DEFINITION 6.12. The *symplectic leaf* of a point on a manifold equipped with a Poisson structure is the set of all points of the manifold that can be reached by paths issuing from the given point, and such that the velocity vectors of the paths are Hamiltonian at every moment (with a Hamiltonian function differentiable in time).

THEOREM 6.13. *The symplectic leaf of every point is a smooth even-dimensional manifold. It has a natural symplectic structure defined by* $\omega(\xi,\eta)=\{f,g\}(x)$, *where* ξ *and* η *are vectors of Hamiltonian fields with Hamiltonian functions* f *and* g *at the point* x.

In particular, the value $\omega(\xi,\eta)$ does not depend on a particular choice of the functions f and g.

On a Poisson manifold the restriction of a Hamiltonian field to each symplectic leaf coincides with the Hamiltonian field defined by the restriction to this leaf of the same Hamiltonian function.

EXAMPLE 6.14. Symplectic leaves of the natural Poisson structure in the dual space to a Lie algebra are group coadjoint orbits. The symplectic structures of the leaves defined by this Poisson structure coincide with the natural symplectic structure of coadjoint orbits described above.

Now let $A:\mathfrak{g}\to\mathfrak{g}^*$ be a nondegenerate symmetric inertia operator. Define the *dual quadratic form* on the dual Lie algebra space $\mathfrak{g}^*$ by

$$H(m):=\frac{1}{2}(A^{-1}m,m), \qquad m\in\mathfrak{g}^*.$$

Denote by v the Lie algebra vector $A^{-1}m$. Then $m=Av$, and therefore

$$H(m)=\frac{1}{2}(v,Av)$$

is merely the kinetic energy, corresponding to the "angular velocity" v (or to the velocity field v in hydrodynamics).

THEOREM 6.15. *Let the inertia operator* A *define a left-invariant metric on the Lie group* G. *Then the Euler velocity field in the dual Lie algebra space* $\mathfrak{g}^*$ *coincides with the Hamiltonian field, defined by the Hamiltonian function* H, *with respect to the natural Poisson structure of the dual Lie algebra. Explicitly, the Euler equation on the dual space* $\mathfrak{g}^*$ *is*

$$\dot m=\mathrm{ad}^*_{A^{-1}m}\,m, \qquad m\in\mathfrak{g}^*. \tag{6.4}$$

For the right-invariant metric the corresponding Hamiltonian function is $-H$.

In particular, the Euler field is a Hamiltonian field on every coadjoint orbit, with respect to the natural symplectic structure of the orbit. Its Hamiltonian function is the restriction of the kinetic energy to the orbit.

PROOF. The differential dH of the quadratic form $H(m) := \frac{1}{2}(A^{-1}m, m)$ at a point $m \in \mathfrak{g}^*$ is the vector $v = A^{-1}m \in \mathfrak{g}$, which is regarded as a linear functional on the dual space $\mathfrak{g}^*$. According to Example 6.10, the Hamiltonian vector corresponding to this linear functional is $\mathrm{ad}^*_{dH}\, m = \mathrm{ad}^*_{A^{-1}m}\, m$, giving Equation (6.4).

The fact that this equation describes the geodesics on the Lie group G with respect to the left-invariant metric is nothing but the Second Euler Theorem.

□

EXAMPLE 6.16. Consider a solution $v(t)$ of the Euler equation of an ideal fluid in a simply connected domain of three-dimensional Euclidean space.

Let $v_1 = v + \varepsilon u_1 + \dots, v_2 = v + \varepsilon u_2 + \dots$ be two solutions with isovorticed initial conditions infinitely close to v (ε is small, the dots mean $o(\varepsilon)$ as $\varepsilon \to 0$, all the fields here and below depend on t).

Isovorticity of the fields means that their vorticities can be identified by a diffeomorphism action. For infinitesimally small perturbations of the vorticities $\xi_i =$ curl u_i we obtain at every moment t

$$\xi_i = [a_i, \mathrm{curl}\ v],$$

where a_i are divergence-free fields from our Lie algebra, and $[\cdot, \cdot]$ is minus the Poisson bracket of the divergence-free fields (so that $[a, b] =$ curl $(a \times b)$).

The fields a_i are not determined uniquely by the perturbations u_i. However, one can define the following invariant of the pair of perturbations that does not depend on this ambiguity.

We associate to the initial field $v(t)$ and to the pair of fields a_i the value

$$\omega = \int\limits_{M^3} ([a_1(t), a_2(t)], v(t))\ dx\, dy\, dz.$$

The theory above, applied to this example, implies the following result.

THEOREM 6.17. *The value of* ω *is constant (i.e., it does not depend on* t*), whatever solution* v *of the Euler equation, and whatever initial fields* $a_1(0)$ *and* $a_2(0)$ *defining the perturbations are taken.*

PROOF. The Hamiltonian property of the Euler equation on the orbits of isovorticed fields implies that the phase flow of the Euler equation preserves the natural symplectic structure of the set of isovorticed fields. This structure is given by the formula $\omega(\xi_1, \xi_2) = (m, [a_1, a_2])$ for $\xi_i = \mathrm{ad}^*_{a_i} m$, where m is the image in the dual space (to the Lie algebra of the divergence-free fields) of the vector $v(t)$ from the Lie algebra under the map $A : \mathfrak{g} \to \mathfrak{g}^*$ of the inertia operator.

According to Theorem 5.4 on inertia operator, the element m can be identified with the 1-form that is the inner product with the vector field v (the dual space itself is understood as the space of 1-forms modulo function differentials).

The differential of the latter 1-form is the vorticity form corresponding to the vorticity vector field $\xi = \mathrm{curl}\, v$. Then the perturbation of the element m defined by the field a_i is

$$\xi_i = [a_i, \xi],$$

and therefore

$$\omega(\xi_1, \xi_2) = \int_{M^3} ([a_1, a_2], v)\; dx\, dy\, dz.$$

The fact that ω does not depend on t means the invariance of the symplectic structure under the Hamiltonian flow of the Euler equation on the coadjoint orbit of the fields isovorticed with $v(0)$. □

REMARK 6.18. On a two-dimensional simply connected domain the fields a_i are defined by the stream functions ψ_i, and the conserved quantity has the form

$$\omega(\xi_1, \xi_2) = \int_{M^2} \xi \cdot \{\psi_1, \psi_2\}\; dx\, dy.$$

Here $\xi_i = \{\psi_i, \xi\}$, $\xi = \Delta\psi$ is the vorticity function of a nonperturbed flow with the stream function ψ, and $\{f, g\}$ is the Poisson bracket of two functions, $\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$, equal to the Jacobian of the map $(x, y) \mapsto (f(x, y), g(x, y))$.

The Hamiltonian property now implies the following. If the vorticity functions ξ and $\xi + \varepsilon\{\psi_i, \xi\} + \dots$ evolve in time according to the Euler–Helmholtz equation, then the value of ω is time invariant. (The Hamiltonian formalism can also be exploited in the reverse direction: from known results in hydrodynamics one can deduce some properties of Hamiltonian systems; see [Ko1].)

We discuss the more general case of Riemannian manifolds, instead of just a domain in Euclidean space, and the non-simply connected domains and manifolds in the next section. There is vast literature on the Hamiltonian formalism of the Euler equation on Lie groups and numerous applications (see, e.g., books [GS2, Arn16, MaR, G-P] and papers [M-W, AKh, Ose2]).

§7. Ideal hydrodynamics on Riemannian manifolds

Generalization of hydrodynamics of ideal incompressible fluid to manifolds of high dimension (in particular, to dimensions $n > 3$) is as physically meaningful as consideration of, say, the wave equation for a non-physically large number of space coordinates. The universal setting, however, sheds light on general properties of the Euler equation, as well as on geometry of the groups of diffeomorphisms. In particular, in this section we will treat the three-dimensional hydrodynamics from this universal point of view.

7.A. The Euler hydrodynamic equation on manifolds. Let M^n denote a compact oriented Riemannian manifold with a metric $(\ ,\ )$ and a volume form μ, i.e., a nonvanishing differential form of the highest degree n. We do not assume, in general, any relation of μ to the volume form induced by the metric.

DEFINITION 7.1. The *Euler equation of an ideal incompressible fluid on M* is the following evolution equation on the velocity field v of the fluid on the manifold:

$$\begin{cases} \frac{\partial v}{\partial t} = -(v, \nabla)v - \nabla p, \\ \mathrm{div}_\mu v = 0, \end{cases} \tag{7.1}$$

where the second equation means that the field v preserves the volume form μ. Here p is a time-dependent function on M that is defined by the condition $\mathrm{div}_\mu(\partial v/\partial t) = 0$ uniquely (up to an additive constant depending on time). The expression $(v, \nabla)v$ denotes the covariant derivative $\nabla_v v$ of the field v along itself for the Riemannian connection on M. In the case of the Euclidean space $M = \mathbb{R}^3$ the Euler equation above assumes the form (5.2).

In the case of a manifold M with boundary, the velocity field is supposed to be tangent to the boundary.

We refer to Section IV.1.B for a definition of the covariant derivative, while for many purposes in this chapter it will be enough to keep in mind the following

EXAMPLE 7.2. In the case of $M = \mathbb{R}^n$, equipped with the standard metric and volume form, the Euler equation of an ideal incompressible fluid is

$$\frac{\partial v_i}{\partial t} = -\sum_{j=1}^{n} v_j \frac{\partial v_i}{\partial x_j} - \frac{\partial p}{\partial x_i}$$

on the vector field v obeying $\sum_{j=1}^{n} \partial v_j/\partial x_j = 0$. The covariant derivative in this case is

$$(v, \nabla)v_i = \sum_{j=1}^{n} v_j \frac{\partial v_i}{\partial x_j}.$$

Just as for the two- and three-dimensional cases, the Euler equation (7.1) on a compact n-dimensional manifold M can be regarded as the equation of geodesics on the Lie group $S\mathrm{Diff}(M)$ of all diffeomorphisms of the manifold M preserving the volume form μ.

DEFINITION 7.3. The *configuration space* of an ideal incompressible fluid filling the manifold M is the *Lie group* $G = S\mathrm{Diff}(M)$ of all diffeomorphisms of M preserving the volume form μ (and belonging to the connected component of the identity). In the case of a manifold with boundary ∂M the group $S\mathrm{Diff}(M)$ consists of those volume-preserving diffeomorphisms that leave the boundary ∂M invariant.

The *Lie algebra* $\mathfrak{g} = S\mathrm{Vect}(M)$ for this group is formed by divergence-free vector fields on M (tangent to the boundary if $\partial M \neq \emptyset$). The Lie bracket in this algebra is *minus* the Poisson bracket of vector fields.

Now we apply the general algebraic machinery to this Lie algebra. The formulations are in Sections 7.B and 7.C below, and the proofs are in Section 8.

7.B. Dual space to the Lie algebra of divergence-free fields. From now on all objects are supposed to be as smooth as needed. We leave aside the analytic difficulties of the approach to infinite-dimensional groups and algebras, and address the interested reader to [E-M], where the proper formalism of the Sobolev spaces for hydrodynamical data is developed. In the sequel we will need the following notions of the calculus on manifolds.

DEFINITION 7.4. Let $\Omega^k(M)$ (or simply Ω^k) denote the space of smooth differential k-forms on the compact manifold M (possibly with boundary ∂M). The *exterior derivative* operator d increases the degree of the forms by 1, while the *inner derivative* operator i_ξ of substitution of a given vector field ξ into a form as the first argument decreases the degree by 1. These operators are derivations of the algebra of forms in the sense that they satisfy the following identities:

$$i_\xi(\alpha \wedge \beta) = (i_\xi \alpha) \wedge \beta + (-1)^k \alpha \wedge (i_\xi \beta), \tag{7.2}$$

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta), \tag{7.3}$$

for any forms $\alpha \in \Omega^k$ and $\beta \in \Omega^l$.

The *Lie derivative* of a differential form ω along a vector field v (tangent to the boundary ∂M if $\partial M \neq \emptyset$) is the time derivative of the form ω transported (backwards) by diffeomorphisms g_t such that $g_0 = \mathrm{Id}$ and $\dot{g}_0 = v$:

$$L_v \omega = \frac{d}{dt}\Big|_{t=0} g_t^* \omega, \tag{7.4}$$

where $g_0(x) \equiv x$, $\frac{d}{dt}\,|_{t=0}\, g_t(x) = v(x)$. The result $g^*\omega$ of the transport of a k-form ω by a smooth map g is defined by the formula

$$(g^*\omega)(\xi_1, \dots, \xi_k) = \omega(g_*\xi_1, \dots, g_*\xi_k),$$

where the linear operator g_* is the differential of the map g.

The *homotopy formula* is the relation

$$L_v = i_v d + d i_v.$$

It is an infinitesimal version of the Leibniz formula: The cylinder boundary is the difference of the top and the bottom, plus the side surface (oriented in the proper way).

THEOREM 7.5 (SEE E.G., [M-W, Nov2, DKN]). *The dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g} = S\,\mathrm{Vect}(M)$ of divergence-free vector fields on M (tangent to ∂M) is naturally isomorphic to the quotient space $\Omega^1/d\Omega^0$ of all differential 1-forms on M, modulo all exact 1-forms (i.e., modulo differentials of all functions) on M.*

The group coadjoint action on the dual Lie algebra $\mathfrak{g}^*$ coincides with the standard action of diffeomorphisms on differential 1-forms:

$$\mathrm{Ad}_g^*\,\alpha = g^*\alpha, \tag{7.5}$$

where 1-forms α and $g^*\alpha$ on M are considered modulo function differentials.

Here we regard $S\,\mathrm{Vect}(M)$ as the Lie algebra of the group of diffeomorphisms of M preserving a fixed volume element. The commutator $\mathrm{ad}_v\, w = [v, w]$ of two vector fields is thus minus their Poisson bracket. We will prove this theorem in Section 8.

COROLLARY 7.6. *The algebra coadjoint action by an element $v \in \mathfrak{g}$ on the dual space $\mathfrak{g}^* = \Omega^1/d\Omega^0$ is the Lie derivative of the 1-forms along the vector field v on M:*

$$\mathrm{ad}_v^*\,\alpha = L_v\alpha. \tag{7.6}$$

Here α and $L_v\alpha$ are 1-forms modulo function differentials.

Indeed, the corollary follows directly from the definition of the Lie derivative (7.4). The infinitesimal version of (7.5) for $g \in G$ close enough to the identity (i.e., for the "infinitesimal change of variables" $g = \mathrm{Id} + \epsilon v + o(\epsilon)$ given by a vector field v) determines the coadjoint action of the Lie algebra element $v \in \mathfrak{g} = S\,\mathrm{Vect}(M)$ on the dual space $\mathfrak{g}^*$ as the derivation along the vector field. □

DEFINITION 7.7. The *pairing* of the spaces $\mathfrak{g}$ and $\mathfrak{g}^*$ is given by the following straightforward formula. Let $[u]$ denote the *coset* of a 1-form u in the quotient $\Omega^1/d\Omega^0$, i.e., the class of all 1-forms on M of the type $u + df$ for some function f. Then, to evaluate a coset $[u] \in \mathfrak{g}^*$ at a vector field $v \in \mathfrak{g}$, one has to take the integral over M of the pointwise pairing of the vector v and an arbitrary 1-form u from the coset $[u] \in \Omega^1/d\Omega^0$:

$$\langle v, [u] \rangle = \int_M u(v)\mu. \tag{7.7}$$

(The fact that this integral does not depend on a particular choice of u is proved in Section 8.) Equivalently, one can think of $\Omega^1/d\Omega^0$ as the space dual to the space of all closed $(n-1)$-forms on M:

$$\langle \omega_v, [u] \rangle = \int_M u \wedge \omega_v, \tag{7.8}$$

where $\omega_v = i_v\mu$ is the closed $(n-1)$-form associated to the divergence-free vector field v.

REMARK 7.8. The group coadjoint action is well-defined by the formula (7.5), since the diffeomorphism action commutes with the operation d: If $\alpha' = \alpha + df$, then

$$g^*\alpha' = g^*\alpha + g^*(df) = g^*\alpha + d(g^*f);$$

i.e., the 1-form $g^*\alpha$ and $g^*\alpha'$ define the same coset in the quotient $\Omega^1/d\Omega^0$. Similarly, the Lie derivative acts on the coset of a 1-form u, since the operation L_v commutes with the derivative operator d:

$$L_v[u] = L_v(u + df) = L_vu + dL_vf = [L_vu].$$

As we discussed above, the space $\mathfrak{g}^*$, in the form of the quotient $\Omega^1/d\Omega^0$, is understood only as the regular part of the actual dual space to the Lie algebra $\mathfrak{g} = S\mathrm{Vect}(M)$. Notice that the nonregular part of the dual space $\mathfrak{g}^*$ includes many interesting functionals, e.g., singular closed 2-forms supported on submanifolds of codimension 2 (for $n = 2$ such forms are supported in a discrete set of points, while for $n = 3$ the support can be a set of closed curves; see Sections I.11 and VI.3).

COROLLARY 7.9. *For a simply connected manifold M (or, more generally, for a manifold with trivial first homology group $H_1(M, \mathbb{R}) = 0$), the dual space $\mathfrak{g}^*$ is isomorphic to the space of all exact 2-forms on M.*

Indeed, the kernel of the operator $d : \Omega^1 \to \Omega^2$ contains all closed 1-forms on M. Simply-connectedness of M (or the condition $H_1(M, \mathbb{R}) = 0$) implies that the first cohomology group vanishes: $H^1(M, \mathbb{R}) = 0$, i.e., all closed 1-forms are exact. Thus the quotient $\Omega^1/d\Omega^0$ is isomorphic to the image of d in Ω^2. □

DEFINITION 7.10. A divergence-free vector field v on M^n (tangent to ∂M) is said to be *exact* if the corresponding closed $(n-1)$-form $\omega_v := i_v\mu$ is the differential of some $(n-2)$-form vanishing on $\partial M : \omega_v = i_v\mu = d\alpha,\ \alpha\mid_{\partial M}= 0$.

EXAMPLE 7.11. On a two-dimensional surface, a field is exact if and only if it possesses a univalued stream function vanishing on the boundary of the surface. The flux of such a field across any closed curve within the surface, as well as the flux across any chord connecting two boundary points, is equal to zero.

On a simply connected manifold of any dimension n every divergence-free vector field is exact. Indeed, due to the identity $H^{n-1}(M) = H^1(M) = 0$, every closed $(n-1)$-form ω_v is exact.

DEFINITION 7.12. A diffeomorphism of a manifold M (preserving the volume element μ and the boundary ∂M) is called *exact* if it can be connected to the identity transformation by a smooth curve g_t (in the space of volume-preserving diffeomorphisms of M) so that the velocity field $\dot{g}_t$ is exact at every moment t. The exact diffeomorphisms constitute the *group of exact diffeomorphisms* $S_0\mathrm{Diff}(M^n)$.

The latter is a subgroup of the group $S\mathrm{Diff}(M^n)$ of all volume-preserving diffeomorphisms. The Lie algebra $\mathfrak{g}_0$ of the group of exact diffeomorphisms $S_0\mathrm{Diff}(M)$ is naturally identified (by the map $v \mapsto i_v\mu$) with the space of differential $(n-1)$-forms, which are differentials of $(n-2)$-forms vanishing on ∂M.

THEOREM 7.13. *The dual space $\mathfrak{g}_0^*$ of the Lie algebra $\mathfrak{g}_0$ for the group $S_0\,\mathrm{Diff}(M)$ is naturally identified with the space of all 2-forms that are differentials of 1-forms on M. This dual space is naturally isomorphic to the quotient $\Omega^1/\ker(d : \Omega^1 \to \Omega^2)$ of the space of all 1-forms on M, modulo all closed 1-forms. The adjoint and coadjoint representations are the standard diffeomorphism actions on $(n-1)$-forms and on 1-forms.*

REMARK 7.14. The *subgroup* generated by the commutators $aba^{-1}b^{-1}$ of a group G is called the *commutant* of G. The commutant of the group $S\mathrm{Diff}(M)$ is the subgroup of the exact diffeomorphisms $S_0\mathrm{Diff}(M)$ [Ban].

The tangent space to the commutant subgroup of a Lie group is called the *commutant subalgebra* of the Lie algebra. It is generated (as a vector space) by the commutators of the Lie algebra elements. The commutant of the Lie algebra $S\mathrm{Vect}(M)$ of the group $S\mathrm{Diff}(M)$ is the Lie algebra $S_0\mathrm{Vect}(M)$ [Arn7].

The quotient space $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ of a Lie algebra $\mathfrak{g}$ by its commutant subalgebra $[\mathfrak{g}, \mathfrak{g}]$ is called the one-dimensional homology (with coefficients in numbers) of the Lie algebra $\mathfrak{g}$. Thus the one-dimensional homology of the Lie algebra of divergence-free vector fields on M^n is naturally isomorphic to the de Rham cohomology group

$$H^{n-1}(M^n, \mathbb{R}) = \ker(d : \Omega^{n-1} \to \Omega^n)/ \operatorname{Im}(d : \Omega^{n-2} \to \Omega^{n-1})$$

(for a manifold with boundary the $(n-1)$-forms have to vanish on the boundary).

The image of a divergence-free vector field $v \in \mathfrak{g} = S\,\mathrm{Vect}(M)$ in the one-dimensional homology group

$$\mathfrak{g}/\mathfrak{g}_0 \approx H^{n-1}(M, \mathbb{R}) \approx H_1(M, \mathbb{R})$$

is called the *rotation class* of a vector field.

The rotation class of a divergence-free vector field concentrated along a closed curve γ in M is the homology class of γ (provided that the flux of the field across a transverse section to γ equals 1).

REMARK 7.15. In the space $C_1(M)$ of closed curves on a manifold M with boundary there are two interesting subspaces: (i) the curves homologous to zero and (ii) the curves in M homologous to those on ∂M. (Recall that two oriented closed curves a, b on M are homologous if there exists a surface S in M whose boundary is $\partial S = a - b$. Here the minus sign means the reversed orientation.)

One defines two subspaces in the space $\mathfrak{g}$ of all divergence-free vector fields on M tangent to ∂M that correspond to the above mentioned subclasses of curves: (i) exact fields $v \in \mathfrak{g}_0$ (such that $i_v\mu = d\alpha,\ \alpha|_{\partial M} = 0$) and (ii) semiexact fields $v \in \mathfrak{g}_{se}$, for which $i_v\mu = d\alpha,\ d\alpha|_{\partial M} = 0$.

THEOREM 7.16. *The subspaces mentioned above are Lie subalgebras in the Lie algebra $\mathfrak{g}$ of divergence-free vector fields on M tangent to the boundary. Moreover, they are Lie ideals; i.e., the Poisson bracket $\{w, v\}$ of an arbitrary field $w \in \mathfrak{g}$ with a field v from either of the subalgebras $\mathfrak{a}$ ($\mathfrak{a} = \mathfrak{g}_0$ or $\mathfrak{g}_{se}$) belongs to the same subalgebra.*

PROOF. A diffeomorphism g from the group $S\mathrm{Diff}(M)$ acts on both the field v and the form α in a consistent way, such that the relations $i_v\mu = d\alpha,\ \alpha|_{\partial M} = 0$ and $d\alpha\,|_{\partial M}$ are preserved under the action. Therefore, every transform Ad_g sends each subalgebra $\mathfrak{a}$ into itself. Let g_t leave the group unity e with velocity $\dot g\,|_{t=0}= w$. The derivative Ad_{g_t} in t takes $\mathfrak{a}$ into itself. This derivative is, up to a sign, the Poisson bracket with the field w. □

THEOREM 7.17. *The dual space $\mathfrak{g}_{se}^*$ (of the Lie algebra $\mathfrak{g}_{se}$ of semiexact divergence-free vector fields) is naturally isomorphic to the quotient space of all 1-forms on M modulo the closed 1-forms on M vanishing on the boundary ∂M.*

7.C. Inertia operator of an n-dimensional fluid.

DEFINITION 7.18. A Riemannian metric $(\ ,\ )$ and a measure μ on the compact manifold M (possibly with boundary ∂M) define a nondegenerate *inner product* $\langle\ ,\ \rangle_{\mathfrak{g}}$ on (divergence-free) vector fields $v, w \in \mathfrak{g}$:

$$\langle v, w\rangle_{\mathfrak{g}} := \int_M (v(x), w(x))\ \mu. \tag{7.9}$$

Hence it specifies an invertible *inertia operator* $A : \mathfrak{g} \to \mathfrak{g}^*$ from the Lie algebra $\mathfrak{g}$ to its dual $\mathfrak{g}^*$ such that the image Av of an element $v \in \mathfrak{g}$ is the element of the dual space $\mathfrak{g}^*$ satisfying

$$\langle Av, w\rangle = \langle v, w\rangle_{\mathfrak{g}} \tag{7.10}$$

for any $w \in \mathfrak{g}$, where $\langle\ ,\ \rangle$ on the left-hand side means the pairing between two elements of the dual spaces. (Strictly speaking, nondegeneracy of the inner product implies invertibility of A only on the regular part of $\mathfrak{g}^*$.)

THEOREM 7.19. *The inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$ for the Lie algebra $\mathfrak{g} = S\,\mathrm{Vect}(M)$ of divergence-free vector fields (tangent to the boundary of M) sends a vector field $v \in \mathfrak{g}$ to the coset $[u] \in \mathfrak{g}^*$ containing the 1-form u obtained from the field v by means of the Riemannian "lifting indices": $u(\xi) = (v, \xi)$ for all $\xi \in T_xM$ at any point $x \in M$.*

PROOF. The theorem is proved by comparison of formulas (7.7) and (7.9). In the tangent space T_xM at every point $x \in M$ the Riemannian "lifting indices" of a vector $v(x)$ is exactly the choice of an exterior 1-form on T_xM whose value on any vector $w(x)$ is the Riemannian inner product of $v(x)$ and $w(x)$. □

For instance, if M is the Euclidean space $\mathbb{R}^n$, the inertia operator sends a vector field $\sum_i v_i(x)\frac{\partial}{\partial x_i}$ to the set of 1-forms $\{\sum_i (v_i(x) + \frac{\partial f}{\partial x_i})dx_i \,\big|\, f \in C^\infty(\mathbb{R}^n)\}$.

Note that the case of a non compact manifold M, say, $M = \mathbb{R}^n$, needs specification of the decay of the vector fields and differential forms at infinity to make the integrals (7.7) and (7.9) converge.

DEFINITION 7.20. The *energy function* on the Lie algebra $\mathfrak{g}$ of divergence-free vector fields is half the square length of vectors $v \in \mathfrak{g}$ in the inner product $\langle\ ,\ \rangle_{\mathfrak{g}}$:

$$H(v) := \frac{1}{2}\langle v, v\rangle_{\mathfrak{g}} = \frac{1}{2}\int_M (v, v)\ \mu = \frac{1}{2}\langle v, Av\rangle.$$

The dual space $\mathfrak{g}^*$ inherits from $\mathfrak{g}$ the nondegenerate inner product $\langle\ ,\ \rangle_{\mathfrak{g}^*}$. We define the *energy Hamiltonian* function on $\mathfrak{g}^*$ as half the square length of the elements in $\mathfrak{g}^*$:

$$H([u]) = \frac{1}{2}\langle [u], [u]\rangle_{\mathfrak{g}^*} := \frac{1}{2}\langle A^{-1}[u], [u]\rangle.$$

Here $v = A^{-1}[u]$ is the (divergence-free) vector field related to the coset $[u]$ of 1-forms by means of the Riemannian metric.

Recall that the Euler equations represent the projection of the geodesic flow of the *right*-invariant metric on the group defined by the quadratic form H on the Lie algebra.

LEMMA–DEFINITION 7.21 [Arn16, OKC]. *The* generalized Euler equation *on the dual space $\mathfrak{g}^* = \Omega^1/d\Omega^0$ of the Lie algebra of divergence-free vector fields on M has the following form:*

$$\frac{\partial [u]}{\partial t} = -L_v[u]. \tag{7.11}$$

Here the vector field v is related to the coset $[u]$ of 1-forms by the metric lifting indices on M: $[u] = Av$. Rewritten for a particular representative 1-form $u \in [u]$, the generalized Euler equation becomes

$$\frac{\partial u}{\partial t} = -L_v u - df. \tag{7.12}$$

The Euler equation on $\mathfrak{g}^*$ is Hamiltonian with respect to the natural Lie–Poisson structure, and minus the energy $-H([u])$ is its Hamiltonian function.

REMARK 7.22. One can see that the latter equation on the dual space $\mathfrak{g}^*$ is the image under the inertia operator A of the classical Euler equation (7.1):

$$\frac{\partial v}{\partial t} = -(v, \nabla)v - \nabla p$$

in the Lie algebra $\mathfrak{g}$ of divergence-free vector fields ($\operatorname{div}_\mu v = 0$). Here the 1-form u is related by metric lifting indices with the vector field v.

The identification of the equations in the Lie algebra and in its dual is based on the following fact, which we shall prove in Section IV.1.D: The inertia operator A sends the covariant derivative vector field $(v, \nabla)v$ on a Riemannian manifold M to the 1-form $L_v u - \frac{1}{2} d(u(v))$. Then the pressure function p is equal to $f + \frac{1}{2} u(v)$ (modulo an additive constant).

The Helmholtz curl equation $\frac{\partial \omega}{\partial t} = -L_v \omega$ on the space of all exact 2-forms $\omega = du$ on M (see Equation (5.3)) is obtained by the exterior differentiation of both sides of Equation (7.11). An advantage of the Helmholtz formulation is the pure geometric action on the 2-forms: the form ω is "frozen into the fluid"; i.e., it is transported by the fluid flow exactly, not just modulo some differential (as is the 1-form u).

COROLLARY 7.23. *If the initial vector field is exact (respectively, semiexact), it will remain exact (respectively, semiexact) for all t.*

This follows from the explicit form of the Euler equation (7.11) and Theorems 7.13 and 7.17 describing the dual spaces to the Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}_{se}$.

PROOF OF LEMMA–DEFINITION. Equation (7.11) is a Hamiltonian equation on $\mathfrak{g}^*$ with minus the energy $-H([u])$ playing the role of the Hamiltonian function.

Indeed, with respect to the standard linear Lie–Poisson structure, the quadratic Hamiltonian function $-H([u]) = -\frac{1}{2}\langle A^{-1}[u], [u]\rangle$ defines the following equation on $\mathfrak{g}^*$:

$$\frac{\partial [u]}{\partial t} = -\mathrm{ad}^*_{A^{-1}[u]}[u],$$

see (6.4). By substituting the explicit form of the inertia operator (Theorem 7.19) and of the coadjoint operator ad* from (7.6), we complete the proof. □

REMARK 7.24. For an arbitrary Lie group G and an arbitrary (not necessarily quadratic) Hamiltonian functional F on the dual space $\mathfrak{g}^*$ to its Lie algebra $\mathfrak{g}$, the corresponding Hamiltonian equation, with respect to the Lie–Poisson structure, is

$$\dot{m} = \mathrm{ad}^*_{\delta F/\delta m}\, m,$$

where the *variational derivative* $\delta F/\delta m$ of the functional F at the point $m \in \mathfrak{g}^*$ is understood as an element of the Lie algebra $\mathfrak{g}$ and is defined by the relation

$$\frac{d}{d\varepsilon}F(m+\varepsilon w)\mid_{\varepsilon=0} = \langle w\ ,\ \delta F/\delta m\rangle,$$

for all $w \in \mathfrak{g}^*$, cf. Example 6.10. For a quadratic functional $F = \frac{1}{2}\langle A^{-1}m,\ m\rangle$ the variational derivative is $\delta F/\delta m = A^{-1}(m)$.

§8. Proofs of theorems about the Lie algebra of divergence-free fields and its dual

Let M^n be a smooth compact manifold with boundary ∂M and volume element μ. Denote by $\mathfrak{g} = S\mathrm{Vect}\,(M)$ the Lie algebra of all divergence-free vector fields on M that are tangent to ∂M. Let $\Omega^k(M)$ be the space of differential k-forms on M and $\Omega^k(M, \partial M)$ the space of differential k-forms on M whose restriction to ∂M vanishes.

To a vector field v on M we associate the following differential $(n-1)$-form:

$$\omega_v = i_v\mu \in \Omega^{n-1}(M).$$

LEMMA 8.1. *The map $v \mapsto w_v$ defines a natural (i.e., invariant with respect to the $S\,\mathrm{Diff}(M)$-action) isomorphism of the vector space of the Lie algebra $\mathfrak{g}$ and the space of all closed differential $(n-1)$-forms on M vanishing on ∂M:*

$$\omega_v \in \ker\left(d : \Omega^{n-1}(M, \partial M) \to \Omega^n(M)\right).$$

PROOF. We start with the fundamental *homotopy formula.*

DEFINITION 8.2. The *Lie derivative operator* L_ξ on forms does not change their degree. It evaluates the instantaneous velocity of the form evolved with the medium whose velocity field is ξ. The linear operator L_ξ is expressed in terms of the operators i_ξ and d via the "homotopy formula" $L_\xi = i_\xi \circ d + d \circ i_\xi$.

Now the proof is achieved by applying the homotopy formula to the volume form $\mu \in \Omega^n(M)$. We conclude that

$$L_v\mu = di_v\mu = d\omega_v,$$

i.e., the flow of v preserves μ if and only if the $(n-1)$-form ω_v is closed. The restriction of ω_v to ∂M is the $(n-1)$-form that gives the flux of the field v across ∂M. The vanishing of ω_v on ∂M is equivalent to the tangency of v to ∂M. □

The statement on duality between the spaces $\mathfrak{g}$ and $\mathfrak{g}^*$ from Sections 3 and 7 has the following precise meaning.

THEOREM 8.3 (SEE ALSO THEOREMS 3.11, 7.5). *For an n-dimensional compact manifold M with boundary ∂M, the dual space $\mathfrak{g}^*$ (of the Lie algebra $\mathfrak{g}$ of divergence-free vector fields on M tangent to ∂M) is naturally isomorphic to the quotient space $\Omega^1(M)/d\Omega^0(M)$ (of all 1-forms on M modulo full differentials) in the following sense:*

(1) *If α is the differential of a function ($\alpha = df$) and $v \in \mathfrak{g}$, then $\iint_M \omega_v \wedge \alpha = 0$.*
(2) *If $\iint_M \omega_v \wedge \alpha = 0$ for all $v \in \mathfrak{g}$, then the 1-form α is the differential of a function.*
(3) *If $\iint_M \omega_v \wedge \alpha = 0$ for all $\alpha = df$, then $v \in \mathfrak{g}$ (i.e., v is a divergence-free field on M tangent to ∂M).*
(4) *The coadjoint action of the group $S\,\mathrm{Diff}(M)$ on the space $\Omega^1(M)/d\Omega^0(M)$ is geometric, i.e., the volume-preserving diffeomorphisms act as changes of coordinates on the (cosets of) 1-forms α.*

PROOF. 1) We utilize the Leibniz identity for the exterior derivative d:

$$d(f \wedge \omega_v) = (df) \wedge \omega_v + f(d\omega_v). \tag{8.1}$$

If $v \in \mathfrak{g}$, then $d\omega_v = 0$ by virtue of Lemma 8.1. Hence

$$\iint\limits_M (df) \wedge \omega_v = \iint\limits_M d(f \wedge \omega_v) = \int\limits_{\partial M} f \wedge \omega_v,$$

according to the Stokes formula. Since $\omega_v \mid_{\partial M} = 0$, the latter integral equals zero.

2) Consider a closed curve γ in M (not meeting the boundary ∂M). Let v be a divergence-free vector field that is supported in a narrow solitorus around this curve and whose flux across any transverse to γ is equal to 1. As the thickness ϵ of the solitorus goes to zero we obtain

$$\lim_{\epsilon \to 0} \iint\limits_M \omega_v \wedge \alpha = (-1)^{n-1} \int\limits_\gamma \alpha = 0$$

for an arbitrary closed curve γ. Therefore, α is the differential of a function (namely, of its integral along a curve connecting the current point with a fixed one).

3) Again, make use of the identity (8.1). Now we know that the integral of $(df) \wedge \omega_v$ over M is equal to zero; hence

$$\iint\limits_M f d\omega_v = \int\limits_{\partial M} f \wedge \omega_v$$

for each function f. Pick a δ-type function f supported in a small neighborhood of an interior point of M. Then the integral on the right-hand side vanishes; hence, the left integral is equal to zero as well. It follows that $d\omega_v = 0$ at every interior point of M; that is, the form ω_v is closed (and the field v is divergence free).

Thus both the left and right integrals are zero for an arbitrary function f. In particular, one can take a function whose restriction to ∂M is of δ-type. At every point of the boundary we obtain $\omega_v|_{\partial M} = 0$. In other words,

$$\omega_v \in \ker\left(d : \Omega^{n-1}(M, \partial M) \to \Omega^n(M)\right),$$

and hence $v \in \mathfrak{g}$ by virtue of Lemma 8.1. Item (3) is proved.

4) The statements (1)–(3) imply that $\mathfrak{g}^* = \Omega^1 / d\Omega^0$, since the set of divergence-free vector fields on M with a volume form μ is identified with the space of closed $(n-1)$-forms by means of the correspondence $v \mapsto \omega_v := i_v \mu$. If M has boundary ∂M, then a field v is tangent to ∂M if and only if the form ω_v vanishes on the boundary.

Furthermore, recall that the adjoint action of the group $S\,\mathrm{Diff}(M)$ on a vector field v is a geometric action (change of coordinates) by a diffeomorphism $g \in G$ on v:

$$\mathrm{Ad}_g\, v = g_* v.$$

It follows that the action of the diffeomorphism g on any 1-form α, which is paired with v is also geometric. More precisely, the group coadjoint action on the coset $[\alpha] \in \Omega^1/d\Omega^0 = \mathfrak{g}^*$ representing the 1-form α in the dual space $\mathfrak{g}^*$ is described as follows:

$$\begin{aligned}\langle v, \mathrm{Ad}_g^*[\alpha]\rangle :=& \langle \mathrm{Ad}_g v, [\alpha]\rangle = \int_M \alpha(g_* v)\ \mu \\ &= \int_M (g^*\alpha)(v)\ g^*\mu = \int_M (g^*\alpha)(v)\ \mu = \langle v, [g^*\alpha]\rangle.\end{aligned}$$

Here we make use of the invariance of the volume form: $g^*\mu = \mu$. Thus

$$\mathrm{Ad}_g^*[\alpha] = [g^*\alpha],$$

which completes the proof of the Theorem. □

Consider now the Lie algebra $\mathfrak{g}_0$ of all exact fields v, for which $\omega_v = i_v\mu = d\beta$ for an $(n-2)$-form $\beta \in \Omega^{n-2}(M, \partial M)$ vanishing on the boundary $\beta|_{\partial M} = 0$.

THEOREM 8.4. *For an n-dimensional compact manifold M with boundary:*

(1) *If the 1-form α is closed ($d\alpha = 0$), then $\iint\limits_M \omega_v \wedge \alpha = 0$ for all fields $v \in \mathfrak{g}_0$.*

(2) *If $\iint\limits_M \omega_v \wedge \alpha = 0$ for all v in $\mathfrak{g}_0$, then the 1-form α is closed in M.*

(3) *If $\iint\limits_M \omega_v \wedge \alpha = 0$ for all closed 1-forms α on M, then $v \in \mathfrak{g}_0$.*

In other words, the dual space $\mathfrak{g}_0^*$ (of the Lie algebra $\mathfrak{g}_0$ of exact divergence-free vector fields) is naturally isomorphic to the quotient space $\Omega^1(M)/Z^1(M)$ (of all 1-forms on M modulo all closed 1-forms on M).

PROOF. 1) Apply the Leibniz identity in the form

$$d(\beta \wedge \alpha) = (d\beta) \wedge \alpha + (-1)^{n-2}\beta \wedge d\alpha, \tag{8.2}$$

where $\omega_v = d\beta$. If $d\alpha = 0$, then, by virtue of the Stokes formula,

$$\iint\limits_M (d\beta) \wedge \alpha = \int\limits_{\partial M} \beta \wedge \alpha = 0,$$

since $\beta\,|_{\partial M} = 0$.

2) By making use of (8.2) when $\iint\limits_M (d\beta) \wedge \alpha = 0$, we obtain

$$(-1)^n \iint\limits_M \beta \wedge d\alpha = \int\limits_{\partial M} \beta \wedge \alpha.$$

The latter integral equals zero for every form β vanishing on ∂M. In particular,

$$\iint_M \beta \wedge d\alpha = 0$$

for every $(n-2)$-form β supported compactly inside M. This implies that $d\alpha = 0$.

3) For a closed form α, we get from (8.2) that

$$0 = \iint_M (d\beta) \wedge \alpha = \iint_{\partial M} \beta \wedge \alpha.$$

Hence, on the closed manifold ∂M the $(n-2)$-form $\beta\,|_{\partial M}$ is orthogonal to every closed 1-form α. By virtue of the Poincaré duality, the $(n-2)$-form $\beta\,|_{\partial M}$ is exact, i.e., there exists an $(n-3)$-form γ on ∂M such that $\beta\,|_{\partial M} = d\gamma$. Extend arbitrarily the $(n-3)$-form γ from ∂M to an $(n-3)$-form $\tilde{\gamma}$ defined on the whole of M (say, extend γ into an ε-neighborhood of ∂M as the pull-back $p^*\gamma$, where p is a retraction to the boundary ∂M, and then multiply the result by a cutoff function equal to 1 in the ε-neighborhood and to 0 outside the 2ε-neighborhood). The restriction of $d\tilde{\gamma}$ to ∂M coincides with $\beta\,|_{\partial M}$. Therefore, $\tilde{\beta} = \beta - d\tilde{\gamma}$ is the required $(n-2)$-form on M that vanishes on ∂M and whose differential is $d\beta = i_v\mu$. Thus, $v \in \mathfrak{g}_0$. □

We leave it to the reader to adjust the above arguments to prove Theorem 7.17.

§9. Conservation laws in higher-dimensional hydrodynamics

The Euler equation of a two-dimensional fluid has an infinite number of conserved quantities (see Section 5). For example, for the standard metric in $\mathbb{R}^2$ one has the *enstrophy invariants*

$$J_k(v) = \int_{\mathbb{R}^2} (\text{curl } v)^k \; d^2x = \int_{\mathbb{R}^2} (\Delta\psi)^k \; d^2x, \qquad \text{for } k = 1, 2, \ldots,$$

where ψ is the "stream function" of the vector field v : $v_1 = -\partial\psi/\partial x_2$, $v_2 = \partial\psi/\partial x_1$.

For an ideal fluid filling a three-dimensional simply connected manifold one has the helicity (or Hopf) invariant, which expresses the mutual linking of the trajectories of the vorticity field curl v, and we discuss it in detail in Chapter III. In the Euclidean space $\mathbb{R}^3$, it has the form

$$J(v) = \int_{\mathbb{R}^3} (v, \; \text{curl } v) \; d^3x.$$

This pattern seems to be rather disappointing. One can hardly expect any first integrals in the higher-dimensional case (except for the energy, of course — the kinetic energy is always invariant being the Hamiltonian function of the Euler equation). It turns out, however, that enstrophy-type integrals do exist for all even-dimensional ideal fluid flows, and so do helicity-type integrals for all odd-dimensional flows. First, we formulate the result for a domain in Euclidean space.

THEOREM 9.1 ([Ser1, Dez] FOR ODD n, [Tar] FOR EVEN n). *The Euler equation (7.1) of an ideal incompressible fluid on a Riemannian manifold in a bounded domain M in $\mathbb{R}^n$ has*

(1) *the first integral*

$$\tilde{I}(v) = \int_M \sum_{(i_1 \dots i_{2m+1})} \varepsilon^{i_1 \dots i_{2m+1}} v_{i_1} \omega_{i_2 i_3} \dots \omega_{i_{2m} i_{2m+1}} \tag{9.1a}$$

if the dimension n is odd: $n = 2m+1$;

(2) *an infinite number of independent first integrals*

$$\tilde{I}_k(v) = \int_M \left(\det \|\omega_{ij}\|\right)^k \; d^n x \tag{9.1b}$$

if the dimension n is even: $n = 2m$.

Here v is the velocity vector field of the fluid in M, the functions $\omega_{ij} := \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}$ are components of the vorticity tensor, $\det\|\omega_{ij}\|$ is the determinant of the skew-symmetric matrix $\|\omega_{ij}\|$, the summation in (9.1a) goes over all permutations of the set $(1 \dots 2m+1)$, and $\varepsilon^{i_1 \dots i_{2m+1}}$ is the Kronecker symbol:

$$\varepsilon^{i_1 \dots i_{2m+1}} = \begin{cases} 1, & \text{if the permutation } (i_1 \dots i_{2m+1}) \text{ of } (1 \dots 2m+1) \text{ is even,} \\ -1, & \text{if the permutation } (i_1 \dots i_{2m+1}) \text{ of } (1 \dots 2m+1) \text{ is odd.} \end{cases}$$

In particular, for $n = 2$ we get from (9.1b):

$$\tilde{I}_k(v) = \int_M \left((\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i})^2 \right)^k \; d^2 x = \int_M \left((\Delta \psi)^2 \right)^k \; d^2 x = J_{2k}(v),$$

while for $n = 3$ the invariant $\tilde{I}(v)$ from (9.1a) assumes the form of the helicity $J(v)$:

$$\tilde{I}(v) = \int_M \sum_{(i_1 i_2 i_3)} \varepsilon^{i_1 i_2 i_3} v_{i_1} \omega_{i_2 i_3} = J(v).$$

Note that in (9.1b) the parameter k is not necessarily an integer.

This theorem follows, practically without calculations, from the definition of the coadjoint action of the diffeomorphisms group when formulated in the invariant and coordinate-free way.

Define the 1-form u as the inner product with the velocity field v in the sense of the Riemannian metric on a manifold M:

$$u(\xi) = (v, \xi) \qquad \text{for all } \xi \in T_x M.$$

THEOREM 9.2 ([OKC, KhC]). *The Euler equation (7.1) of an ideal incompressible fluid on a Riemannian manifold M^n (possibly with boundary) with a measure form μ has*

(1) *the first integral*

$$I(v) = \int_M u \wedge (du)^m \tag{9.2a}$$

in the case of an arbitrary odd-dimensional manifold M $(n = 2m+1)$; and

(2) *an infinite number of functionally independent first integrals*

$$I_f(v) = \int_M f\left(\frac{(du)^m}{\mu}\right)\mu \tag{9.2b}$$

in the case of an arbitrary even-dimensional manifold M $(n = 2m)$,

where the 1-form u and the vector field v are related by means of the metric on M, and $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function of one variable.

The fraction $(du)^m/\mu$ for $n = 2m$ is a ratio of two differential forms of the highest degree n. Since the volume form μ vanishes nowhere, the ratio is a well-defined function on M (which may depend on time t). The integral of the function f evaluated at this ratio gives a generalized momentum (i.e., a weighted volume between different level hypersurfaces) of the invariant function $(du)^m/\mu$. The momenta $\tilde{I}_k$ correspond to the choice $f(z) = z^{2k}$. Theorem 9.1 can be obtained from Theorem 9.2 by coordinate rewriting of the differential 2-form du as the matrix $\|\omega_{ij}\|$.

PROOF. The trajectories of the Euler equation on $\mathfrak{g}^*$ belong to the coadjoint orbits of the group G. This immediately follows from the Hamiltonian formulation of the Euler equation: The trajectories belong to the symplectic leaves of the Lie–Poisson bracket on $\mathfrak{g}^*$, which are the coadjoint orbits of G.

Now the invariance of the functionals I and I_f along the trajectories follows from

PROPOSITION 9.3. *The following functionals on $\mathfrak{g}^*$ are invariants of the coadjoint action:*

(1) *in case $n = 2m + 1$*

$$I([u]) = \int_M u \wedge (du)^m,$$

(2) *in case $n = 2m$*

$$I_f([u]) = \int_M f\left(\frac{(du)^m}{\mu}\right)\mu,$$

where f is an arbitrary function of one variable, and $[u] \in \Omega^1/d\Omega^0 = \mathfrak{g}^*$ is the coset of a differential 1-form u.

PROOF OF PROPOSITION. The above functionals are well-defined on $\mathfrak{g}^*$, i.e., they do not depend on the ambiguity in the choice of the representative 1-form u. Indeed, under a change of u to another representative $u+dh$ in the same coset $[u] \in \Omega^1/d\Omega^0$ the form du will not be affected. Hence, the invariants I_f rely merely on the coset $[u]$ of the form u, and so does I, since $I(u+dh) - I(u) = \int dh \wedge (du)^m = 0$.

The coadjoint action of the diffeomorphism group G coincides with the change of variables (see Theorem 7.5) in 1-forms u (or in the corresponding cosets). The integrals I and I_f are defined in a coordinate-free way; hence, they are invariant under the coadjoint action. This completes the proof of Proposition 9.3, as well as of the two preceding theorems. □

REMARKS 9.4. A) At first glance, it seems that one can generate more invariants in odd dimensions by considering shear plane-parallel flows of one dimension higher and using the corresponding even-dimensional invariants. However, the reduction from even to odd dimensions does not provide any new integrals different from (9.2a). The reason is that the invariant (9.2b) for a shear plane-parallel $2m$-dimensional flow obtained from a $(2m-1)$-dimensional one is trivial: A plane-parallel vector field v induces the 2-form du of rank less than $2m$, since the additional direction lies in the kernel of du. This implies that $(du)^m = 0$, and the corresponding integrals (9.2b) become trivial.

B) For a noncompact manifold M (say, for the whole space $\mathbb{R}^n$), we should confine ourselves to the class of vector fields and forms decaying fast enough to make convergent the above integrals over M.

The manifold M may be multiconnected. In the case of a non-simply connected manifold M, the cohomology class of the 1-form u (or of the coset $[u]$) corresponding to the vector field v is also invariant (cf. [Arn7]). Other examples of first integrals of the Euler equation are provided by the number of points or submanifolds in M where the two-form du is degenerate, as well as by the orders of its degeneracy there, and by the invariants of the periodic orbits of the velocity field in the three-dimensional case (periods, Floquet multipliers, etc.).

See also [Ol, Gur] for a discussion of the symmetries, i.e., infinitesimal transformations in the jet spaces, preserving the Euler equations for $n = 2$ and $n = 3$.

A natural by-product of the invariant approach to higher-dimensional hydrodynamics is the following notion of vorticity in n dimensions.

DEFINITION 9.5. The *vorticity form* (or *curl*) of a vector field v on an n-dimensional manifold M is the 2-form $\omega = du$ that is the differential of the 1-form u related to v by means of the chosen Riemannian metric. Depending on the parity of the dimension of M, one can associate to the 2-form ω a vorticity function or a vorticity vector field.

On an even-dimensional manifold M^n $(n = 2m)$ the ratio $\lambda = (du)^m/\mu$ is called the *vorticity function* of the field v.

On an odd-dimensional manifold M^n $(n = 2m+1)$, the 2-form $\omega = du$ is always degenerate, and the *vorticity vector field* is the kernel vector field ξ of the vorticity form ω: $i_\xi \mu = \omega^m$.

EXAMPLE 9.6. In Euclidean space $\mathbb{R}^{2m}$ with standard volume form, the vorticity function of a vector field v is

$$\lambda = \sqrt{\det \|\omega_{ij}\|},$$

and for $n = 2m = 2$ it is the standard definition of the vorticity function

$$\text{curl } v = \partial v_1/\partial x_2 - \partial v_2/\partial x_1.$$

In $\mathbb{R}^{2m+1}$ with the Euclidean metric the vorticity field ξ has the coordinates

$$\xi_j = \sum_{(j\, i_1 \dots i_{2m})} \varepsilon^{j\, i_1 \dots i_{2m}} \omega_{i_1 i_2} \dots \omega_{i_{2m-1} i_{2m}},$$

where $\varepsilon^{j\, i_1 \dots i_{2m}}$ is the Kronecker symbol, and the summation is over all permutations of $(1 \dots 2m+1)$. In $\mathbb{R}^3$ this expression gives the classical definition of the vorticity field $\xi = \text{curl } v$.

PROPOSITION 9.7. *The vorticity vector field ξ and the vorticity function λ are transported by the Euler flow on, respectively, odd- or even-dimensional manifolds.*

PROOF. Indeed, the coadjoint action is geometric, and it changes coordinates in the 2-form du. Thus du is transported by the flow, while the volume form μ is invariant under it. Hence, the vorticity vector field and function, defined in terms of these two objects, are transported by the incompressible flow as well. □

The above statement is based on the Helmholtz evolution equation valid for the 2-form $\omega = du$: $\frac{\partial \omega}{\partial t} = -L_v \omega$. It means that the substantial derivative of ω vanishes, or that this 2-form is transported by the flow.

REMARK 9.8. The above integrals are invariants of the coadjoint representation of the corresponding Lie groups (the so-called *Casimir functions*), i.e., they are invariants of the Hamiltonian equations with respect to the Lie–Poisson structure on $\mathfrak{g}^*$ for an arbitrary choice of the Hamiltonian function. The integrals $I([u])$ and $I_f([u])$ do not form a complete set of continuous invariants of coadjoint orbits. One can construct parametrized families of orbits with equal values of these functionals, similar to those described in Section 5 for $n = 2$ and in Chapter III for $n = 3$. For instance, in odd dimensions the flow preserves not only the integral (9.2a) over the entire manifold M, but also the integrals

$$I_C(v) = \int_C u \wedge (du)^m$$

over every invariant set C of the vorticity vector field for the instantaneous velocity v. This follows immediately from the Stokes formula, applied to such an invariant set, and from the observation that the restriction of $(du)^m$ to the boundary of any invariant set vanishes.

A precise description of the coadjoint orbits for the diffeomorphism groups still remains an unsolved and intriguing problem. In particular, one may think that the closure of a coadjoint orbit for $n = 3$ could contain an open part of a level set of the integral $I(v)$ in some topology. Physically, this would mean that in the three-dimensional case preservation of vorticity is not as restrictive on the particles' permutations realized by the flow as it is in the two-dimensional case.

The reason for this conjecture is the following result on *local invariants* of the coadjoint orbits (i.e., the local description of isovorticed fields) in ideal hydrodynamics.

THEOREM 9.9.

(1) *The vorticity function λ is the only local invariant of the coadjoint orbits of the group of volume-preserving diffeomorphisms of an even-dimensional manifold M^n ($n = 2m$) at a generic point.*

(2) *For odd-dimensional M^n ($n = 2m + 1$) there are no local invariants of the coadjoint orbits of the group $S\,\mathrm{Diff}(M^n)$; i.e., at a generic point of the manifold the cosets belonging to different coadjoint orbits can be identified by means of a volume-preserving diffeomorphism.*

For instance, in the two-dimensional case $n = 2$, the set of isovorticed fields is fully described by their vorticity function curl $v = \partial v_1/\partial x_2 - \partial v_2/\partial x_1$. In the three-dimensional case $n = 3$, the vorticity vector field can be rectified in the vicinity of every nonzero point by a volume-preserving change of coordinates, and hence has no local invariants. The coadjoint invariant for the odd-dimensional case, provided by Proposition 9.3(2), has genuine global nature: It expresses the linking of vortex trajectories in the manifold; see Chapter 3.

PROOF. In the coordinate-free language, the local invariants of the coadjoint action are associated to a coset $[u] \in \mathfrak{g}^*$ of 1-forms in a small neighborhood on a manifold equipped with a volume form. The local invariants of the coset $[u] = \{u + df\}$ (i.e., a 1-form u up to addition of the function differential) are the same as the local invariants of the 2-form du, since taking the differential of the 1-form kills the ambiguity df.

Therefore, the problem reduces to the description of invariants of a closed 2-form in the presence of a volume form μ, i.e., the form of degree n. If n is even, one can think of du as a symplectic form. The pair (du, μ) has the following invariant function associated to them: the symplectic volume $\lambda = (du)^m/\mu$, where $n = 2m$. This volume is nothing but the vorticity function. The uniqueness of this invariant in a generic point immediately follows from the Darboux theorem: By a (non-volume-preserving) change of variables the 2-form du transforms to $\sum_i dp_i \wedge dq_i$, while the volume form becomes $d^m p\, d^m q/\lambda(p, q)$; see [A-G].

If n is odd, in a generic point there is no invariant for the pair (du, μ). Indeed, a nondegenerate 2-form du again transforms to $du = \sum_{i=1}^m dp_i \wedge dq_i$ in $\mathbb{R}^{2m+1} = \{(p, q, z)\}$, according to a version of the Darboux theorem. Then by changing the coordinate $z \to z' = h(p, q)$ one can reduce the volume form to $\mu = d^m p \wedge d^m q \wedge dz$ without further changing du. Thus a generic pair (du, μ) has a unique canonical form. □

The above theorem does not imply that other invariants that are *integrals of local densities* over the flow domain could not exist. We conjecture that there are no new integral invariants either for the Euler equation or for the coadjoint orbits of the diffeomorphism groups. The *integral invariants* of a vector field v are functionals of the form $\int_M f(v)\, d^n x$. The density function f is called *local* if it depends on only a finite number of partial derivatives of v.

REMARK 9.10. The Casimir functions, i.e., invariants of the coadjoint representation, allow one to study the nonlinear stability problems by Routh-type methods (see Chapter II). The information about the orbits can be helpful in the study of the Cauchy problem in high-dimensional hydrodynamics. The different number of invariants in odd and even dimensions apparently indicates that the existence theorem in odd- and even-dimensional hydrodynamics should require essentially different arguments.

Instead of writing the Euler equation as an evolution of a coset $[u]$, one can choose the special 1-form $\bar{u}$ for which the action of the flow is geometric. This would allow one to write the invariants in odd dimensions in the same way as for $n = 2m$, since, say, the ratio $\bar{u} \wedge (d\bar{u})^{2m}/\mu$ is transported by the flow for such a choice of $\bar{u}$. To find the corresponding evolution of $\bar{u}$ one has actually to solve the Euler equation for the velocity field [Ose2, GmF]. Such invariants are similar to Lagrangian coordinates of fluid particles.

Note that the existence of an infinite series of integrals for a flow of an ideal even-dimensional fluid does not imply complete integrability of the corresponding hydrodynamic equations. These invariants merely specify the coadjoint orbits (generally speaking, infinite-dimensional) where the dynamics takes place. For the evolution on the orbit itself we know just the energy integral, while integrability requires specification of an infinite number of integrals.

On the other hand, the Euler hydrodynamic equations in the plane admit finite-dimensional truncations of arbitrarily large size that turn out to be integrable Hamiltonian systems [MuR]. We discuss finite-dimensional approximations of classical hydrodynamic equations in Section 11. In Section VI.3 we will show how knot theory can be regarded as a part of coadjoint orbit classification for the group $S\,\mathrm{Diff}(M^3)$. Knots correspond to highly degenerate orbits of differential 2-forms supported on curves in a three-dimensional manifold M. Knot invariants with respect to isotopies become Casimir invariants for such degenerate orbits.

§10. The group setting of ideal magnetohydrodynamics

Magnetic fields in perfectly conducting plasma or magma are among the main objects of study in astrophysics and geophysics. In the idealized setting, an inviscid incompressible fluid obeying hydrodynamical principles transports a magnetic field. In turn, the medium itself experiences a reciprocal influence of the magnetic field. The evolution is described by the corresponding system of Maxwell's equations.

10.A. Equations of magnetohydrodynamics and the Kirchhoff equations.

Definition 10.1. We assume first that an electrically conducting fluid fills some domain M of the Euclidean three-dimensional space $\mathbb{R}^3$. The fluid is supposed to be incompressible with respect to the standard volume form $\mu = d^3x$, and it transports a divergence-free magnetic field $\mathbf{B}$. Then, the evolution of the field $\mathbf{B}$ and of the fluid velocity field v is described by the system of ideal *magnetohydrodynamics (MHD) equations*

$$\begin{cases} \frac{\partial v}{\partial t} = -(v,\nabla)v + \ (\mathrm{curl}\ \mathbf{B}) \times \mathbf{B} - \nabla p, \\ \frac{\partial \mathbf{B}}{\partial t} = -\{v, \mathbf{B}\}, \\ \mathrm{div}\ \mathbf{B} = \mathrm{div}\ v = 0. \end{cases} \tag{10.1}$$

Here the second equation is the definition of the "frozenness" of the magnetic field $\mathbf{B}$ into the medium, and $\{\ ,\ \}$ denotes the Poisson bracket of two vector fields. In the first equation the pressure term ∇p is uniquely defined by the condition $\operatorname{div} \partial v/\partial t = 0$, just as it is for the Euler equation in ideal hydrodynamics. The term $(\operatorname{curl} \mathbf{B}) \times \mathbf{B}$ represents the Lorentz force. On a unit charge moving with velocity $\mathbf{j}$ in the magnetic field $\mathbf{B}$ there acts the Lorentz force $\mathbf{j} \times \mathbf{B}$. On the other hand, the electrical current field $\mathbf{j}$ is equal to $\operatorname{curl} \mathbf{B}/4\pi$ according to Maxwell's equation [Max]. The coefficients in Equation (10.1) are normalized by a suitable choice of units.

The *total energy* E of the MHD system is the sum of the kinetic and magnetic energy:

$$E := \frac{1}{2}\langle v, v\rangle + \frac{1}{2}\langle \mathbf{B}, \mathbf{B}\rangle. \tag{10.2}$$

REMARK 10.2. One can view the *Kirchhoff equations* [Kir]

$$\begin{cases} \dot p = p \times \omega, \\ \dot m = m \times \omega + p \times u \end{cases} \tag{10.3}$$

for a rigid body moving in a fluid as a finite-dimensional analogue of the magnetohydrodynamics (just as the classical rigid body with a fixed point is analogous to the ideal fluid dynamics); see [V-D, DKN]. The fluid is ideal, incompressible, and at rest at infinity, and the fluid motion itself is supposed to be potential. The energy of a body in a fluid is

$$H = \frac{1}{2}\left(\sum a_i m_i^2 + \sum b_{ij}(p_i m_j + m_i p_j) + \sum c_{ij} p_i p_j\right). \tag{10.4}$$

The variables m and p are the total *angular momentum* and the *vector momentum* of the body–fluid system in a moving coordinate system rigidly attached to the body; $u^i = \partial H/\partial p_i$, $\omega^i = \partial H/\partial m_i$. The energy is quadratic in m, p, it is assumed to be positive; and it defines a Riemannian metric on the group $E(3)$ of all motions in three-dimensional Euclidean space.

In the case of magnetohydrodynamics the total energy is to be considered as the Riemannian metric on the configuration space, which is the *semidirect product* of the diffeomorphism group $S\mathrm{Diff}(M)$ and the dual space $\mathfrak{g}^* = \Omega^1(M)/d\Omega^0(M)$. This space and metric are defined below.

10.B. Magnetic extension of any Lie group. Consider the following example: the one-dimensional Lie group G of all dilations of a real line $x \mapsto bx$. The composition of two dilations with factors b_1 and b_2 defines the dilation with the factor $b_1 b_2$. We will call the two-dimensional group F of all affine transformations of the line $x \mapsto a+bx$ the *magnetic extension* of the group G. Now, the composition of two affine transformations $x \mapsto a_1 + b_1 x$ and $x \mapsto a_2 + b_2 x$ sends every point x to the point

$$a_2 + b_2(a_1 + b_1 x) = (a_2 + a_1 b_2) + (b_1 b_2)x.$$

Hence the group multiplications of the pairs (a, b) that constitute the magnetic extension group F, is

$$(a_2,\, b_2) \circ (a_1,\, b_1) = (a_2 + a_1 b_2,\, b_1 b_2);$$

see also Section IV.1.A. The general description of magnetic extensions below can be regarded as a group formalization of this construction.

Let G be an arbitrary Lie group. We associate to this group a new one, called the *magnetic extension* of the group G, in the following way. The elements of the new group are naturally identified with all points of the phase space T^*G whose configuration space is G.

The group G acts naturally on itself by left translations, as well as by right ones. The left and right shifts commute with each other. Hence, right-invariant vector (or covector) fields are taken to right-invariant ones under left translations, while left-invariant fields are sent to left-invariant ones by right translations.

Extend every covector on G, i.e., an element α_g of the cotangent bundle T^*G at $g \in G$, to the right-invariant section (covector field) α on the group. Define the action of this covector α_g on the phase space T^*G as follows. First add to every covector in T^*G at h the value of the right-invariant section α at h. Then apply the left shift of the entire phase space T^*G by g (Fig.8).

THEOREM 10.3. *The result of two consecutive applications of two cotangent vectors of the group coincides with the action of a new cotangent vector. This composition makes the space T^*G into a Lie group.*

PROOF. The composition of two left shifts on a group is a left shift as well. The operator T_2 of addition of the second right-invariant covector field after the first left shift L_1 coincides with the addition T_1 of another right-invariant covector field preceding the first left shift. Namely, the new covector field is the image of the second covector field under the action of the inverse L_1^{-1} of the first left shift L_1:

$$L_2 T_2 L_1 T_1 = L_2 L_1 \tilde{T}_2 T_1,$$

where $\tilde{T}_2 = L_1^{-1} T_2 L_1$. The sum of this new field with the first right-invariant field is the right-invariant covector field that is to be added to each covector of T^*G

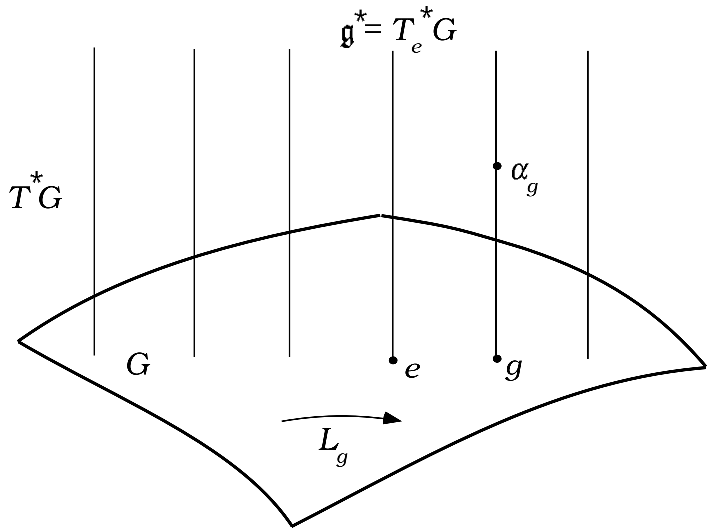

FIGURE 8. The cotangent bundle T^*G turns into a Lie group.

before the left translation–composition to obtain the result of the two consecutive applications of two cotangent vectors. □

Note that a right-invariant field is determined by its value at the group identity. Hence, the phase space T^*G is diffeomorphic to the direct product $G \times \mathfrak{g}^*$ of the group G and of the dual space $\mathfrak{g}^*$ to its Lie algebra (the cotangent bundle of any Lie group is naturally trivialized). However, the *group* T^*G constructed above *is not the direct product* of the group G and the commutative group $\mathfrak{g}^*$.

Consider a group element $(\psi, b) \in T^*G$, i.e., the composition of the addition of a right-invariant field whose value at the group identity is some covector b followed by the left shift by ψ, and similarly, another element $(\phi, a) \in T^*G$.

THEOREM 10.3′. *The composition of* (ψ, b) *followed by* (ϕ, a) *is the left shift by* $\phi \circ \psi$ *preceded by adding the right-invariant field whose value at the identity is the covector* $\mathrm{Ad}^*_\psi a + b$.

PROOF. The left translation by ψ^{-1} of the right-invariant field generated by a at the identity is the right-invariant field whose value at the identity is

$$L^*_\psi R^*_{\psi^{-1}} a = (R_{\psi^{-1}} L_\psi)^* a = (\mathrm{Ad}_\psi)^* a.$$

□

DEFINITION 10.4. The *magnetic extension* $F = G \ltimes \mathfrak{g}^*$ of a group G is the group of pairs $\{(\phi, a) \big| \ \phi \in G, \ a \in \mathfrak{g}^*\}$ with the following group multiplication between the pairs:

$$(\phi, a) \circ (\psi, b) = (\phi \circ \psi, \operatorname{Ad}^*_\psi a + b). \tag{10.5}$$

The definition of the Lie algebra corresponding to the magnetic extension F follows immediately.

DEFINITION 10.5. The Lie algebra $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{g}^*$, corresponding to the (magnetic extension) group $F = G \ltimes \mathfrak{g}^*$, is the vector space of pairs $(v \in \mathfrak{g},\ a \in \mathfrak{g}^*)$ endowed with the following Lie bracket:

$$[(v, a), (w, b)] = ([v, w], \operatorname{ad}^*_w a - \operatorname{ad}^*_v b), \tag{10.6}$$

where $[v, w]$ is the commutator of the elements v and w in the Lie algebra $\mathfrak{g}$ itself, and $\operatorname{ad}^*_w a$ is the coadjoint action of the algebra $\mathfrak{g}$ on its dual space $\mathfrak{g}^*$.

The magnetic extension is a particular case of notion of a *semidirect product* of a Lie group G, or a Lie algebra $\mathfrak{g}$, by a vector space V, where this group or algebra acts. In the general situation the operators Ad and ad of the coadjoint action in (10.5–6) are to be replaced by the action of the corresponding group or algebra elements on the vector space V; see Section VI.2 and [MRW].

EXAMPLE 10.6. The symmetry group $E(3)$ of a rigid body in a fluid is the magnetic extension $E(3) = SO(3) \ltimes \mathbb{R}^3$ of the group $SO(3)$ of all rotations of the three-dimensional space by the dual space $\mathfrak{so}(3) = \mathbb{R}^3$. As we shall see in the next Section, the Kirchhoff equations (10.3) describe the geodesics on this group $E(3)$ with respect to the *left*-invariant metric defined by the energy E from (10.2).

EXAMPLE 10.7. The configuration space of magnetohydrodynamics is the magnetic extension $F = S\mathrm{Diff}(M) \ltimes (\Omega^1/d\Omega^0)$ of the group $G = S\mathrm{Diff}(M)$ of volume-preserving diffeomorphisms of a manifold M and of the corresponding dual space $\mathfrak{g}^* = \Omega^1/d\Omega^0$. The group coadjoint action $\operatorname{Ad}^*_\psi a = \psi_* a$ in (10.5) is the change of coordinates by the diffeomorphism ψ in the coset a of 1-forms. The corresponding operator ad in (10.6) of the coadjoint action of the Lie algebra is the Lie derivative operator on the cosets: $\operatorname{ad}^*_w a = L_w a$.

The MHD equations (10.1) are the geodesic equations on the group F with respect to the *right*-invariant metric defined by the magnetic energy E (10.4); see Theorem 10.9 below.

REMARK 10.8. The above definitions can be applied to any manifold M (of arbitrary dimension) equipped with a volume form. Correspondingly, one can define the equations of magnetohydrodynamics on the manifold once one specifies a Riemannian metric on M whose volume element is the given volume form. The only operation that has not yet been specified in the general setting is the cross product, and this can be done using the isomorphism $*$ of k- and $(n-k)$-polyvector fields induced by the metric on a manifold M of any dimension n [DFN]. We refer to [M-W, KhC] for generalizations of the MHD formalism to other dimensions.

Notice also that in the two-dimensional case one has two options for generalizations of equations (10.1): The magnetic field $\mathbf{B}$ can be regarded as a divergence-free vector field, or, alternatively, as a closed two-form on M. The latter is the same as a function on the two-dimensional manifold M. According to these two possibilities, one has two different systems of equations (see, e.g., the Hamiltonian formulations of MHD presented in [MoG, H-K, ZeK, Ze2]).

10.C Hamiltonian formulation of the Kirchhoff and magnetohydrodynamics equations.

THEOREM 10.9 [V-D, MRW].

(1) *The equations of the magnetic hydrodynamics (10.1) are Hamiltonian equations on the space $\mathfrak{f}^*$ dual to the Lie algebra $\mathfrak{f} = S\,\mathrm{Vect}(M) \ltimes (\Omega^1/d\Omega^0)$*

(2) *The Kirchhoff equations (10.3) are Hamiltonian equations on the space $\mathfrak{e}(3)^*$ dual to the Lie algebra $\mathfrak{e}(3) = \mathfrak{so}(3) \ltimes \mathbb{R}^3$*

relative to the standard Lie–Poisson bracket. The Hamiltonian functions are the quadratic forms on the dual spaces defined by the total energy $-E$ or H (formulas (10.2) and (10.4), respectively).

PROOF. Consider the MHD system in three dimensions. The dual Lie algebra $\mathfrak{f}^*$ of the magnetic extension F as a set of pairs is $\mathfrak{f}^* = \{([u], \mathbf{B})|\ [u] \in \mathfrak{g}^* = \Omega^1/d\Omega^0,\ \mathbf{B} \in \mathfrak{g} = S\mathrm{Vect}(M)\}$. The explicit formula for the coadjoint action of the Lie algebra $\mathfrak{f}$ on its dual space $\mathfrak{f}^*$ is

$$\mathrm{ad}^*_{(v,[\alpha])}([u], \mathbf{B}) = (L_v[u] - L_{\mathbf{B}}[\alpha], -L_v\mathbf{B}). \tag{10.7}$$

Here $-L_v\mathbf{B} = \{v, \mathbf{B}\}$ is the Poisson bracket of the two vector fields.

The Riemannian metric on M defines the isomorphism $A : \mathfrak{g} \to \mathfrak{g}^*$ between the space $\mathfrak{g} = S\mathrm{Vect}(M)$ of divergence-free vector fields and the dual space $\mathfrak{g}^* = \Omega^1/d\Omega^0$; see Section 7. It induces the inner product on the magnetic extension algebra $\mathfrak{f}$, as well as on its dual space $\mathfrak{f}^*$. The corresponding quadratic form of the energy E on $\mathfrak{f}^*$ is

$$E([u], \mathbf{B}) = \frac{1}{2}\langle [u], A^{-1}[u]\rangle + \frac{1}{2}\langle \mathbf{B}, A(\mathbf{B})\rangle.$$

Thus the Euler equation on $\mathfrak{f}^*$ with the Hamiltonian function $-E$, i.e., the geodesic equation for the corresponding right-invariant metric on the group F, is

$$\begin{cases} \frac{\partial [u]}{\partial t} = -L_v[u] + L_{\mathbf{B}}[b], \\ \frac{\partial \mathbf{B}}{\partial t} = -\{v, \mathbf{B}\}, \end{cases} \tag{10.8}$$

where the vector field v and the coset $[b]$ are, respectively, related to the coset $[u]$ and the magnetic vector field $\mathbf{B}$ by means of the inertia operator:

$$v = A^{-1}[u], \qquad [b] = A(\mathbf{B}).$$

Equations (10.1) are equivalent to their intrinsic form (10.8), as the following statement shows.

LEMMA 10.10. *The operator A^{-1} of the Riemannian identification of divergence-free vector fields and the cosets of 1-forms on the manifold M takes the coset $L_{\mathbf{B}}[b]$ (i.e., $L_{\mathbf{B}}A(\mathbf{B})$) to the field* curl $\mathbf{B} \times \mathbf{B}$*, provided that the volume form μ is defined by the Riemannian volume element on M.*

The verification of the latter in a local coordinate system is straightforward.

□

COROLLARY 10.11. *The inner product*

$$J(v, \mathbf{B}) = \int_M (v, \mathbf{B})\ \mu$$

of the fluid velocity and the evolved magnetic field is the first integral of the motion defined by the MHD Equation (10.1).

A way to prove this statement is to differentiate the quantity J along the vector field given by (10.1). It is, however, a consequence of the following, more general, observation.

COROLLARY 10.11′ [V-D]. *Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. Then the quadratic form*

$$J(u, \alpha) = \langle u, \alpha \rangle,$$

on the dual space $\mathfrak{f}^* = \{(u, \zeta) |\ u \in \mathfrak{g}^*,\ \zeta \in \mathfrak{g}\}$ of the magnetic Lie algebra $\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{g}^*$ is an invariant of the coadjoint representation of the Lie group $F = G \ltimes \mathfrak{g}^*$. Here $\langle u, \alpha \rangle$ stands for the pairing of two elements of the dual spaces $\mathfrak{g}$ and $\mathfrak{g}^*$.

PROOF. The invariance of the quadratic form J is verified by direct calculation using the operators of coadjoint action of the group F.

Applying it to the MHD group $F = S\,\mathrm{Diff}(M) \ltimes \Omega^1/d\Omega^0$, we prove the invariance of $J(v, \mathbf{B})$ on the coadjoint orbits of F. The conservation of $J(v, \mathbf{B})$ on the trajectories of (10.1) follows from the Hamiltonian formulation of the equations: The trajectories are tangent to the coadjoint orbits of the group F. □

REMARK 10.12. The quadratic form $J(v, \mathbf{B}) = \int_M u(\mathbf{B})\mu$ (called *cross helicity*) has a simple topological meaning, being the asymptotic linking number of the trajectories of the magnetic field $\mathbf{B}$ with the trajectories of the vorticity field curl v. It is similar to the total *helicity* of an ideal fluid (which measures the asymptotic mutual linking of the trajectories of the fluid vorticity field) or the magnetic helicity (measuring the linking of magnetic lines); see Chapter III.

The vorticity field of an ideal incompressible fluid is transported (convected) by the fluid flow, and topological invariants of the field are preserved in time. However, unlike the helicity in hydrodynamics, the conservation of the mutual linking between the magnetic and vorticity fields in MHD flow is somewhat unexpected, since curl v in magnetohydrodynamics is not frozen (in contrast to the magnetic field $\mathbf{B}$). The evolution changes the field v (and hence curl v as well) by some additive summand, which depends on $\mathbf{B}$, but it turns out that the mutual linking of the vorticity field curl v and the magnetic field $\mathbf{B}$ is preserved (see [VlM] for more detail).

It would be of special interest to find a description of Casimir functions for magnetohydrodynamics. In particular, one wonders whether there exists an MHD analogue of the complete classification of local invariants of the coadjoint action for ideal hydrodynamics (Theorem 9.9) and what are the integral invariants defined by local densities.

§11. Finite-dimensional approximations of the Euler equation

The effort to give a comprehensive finite-dimensional picture of hydrodynamical processes has a long history: Any attempt to model the Euler equation numerically leads to some kind of truncation of the continuous structure of the equation in favor of a discrete analogue.

According to the main line of this book, we will concentrate on the methods preserving the Hamiltonian structure of the Euler equation and will leave aside numerous (and equally fruitful) methods related to difference schemes or to series expansions of the solutions. We discuss the Galerkin-type approximations for solutions of the Navier–Stokes equation in the next section.

11.A. Approximations by vortex systems in the plane. For numerical purposes one usually starts with the Euler equation written in the Helmholtz form

$$\dot{\mathbf{w}} = -\{\mathbf{v}, \mathbf{w}\}, \tag{11.1}$$

which describes the evolution of the vorticity field $\mathbf{w} = \operatorname{curl} \mathbf{v}$ frozen into a flow with velocity $\mathbf{v}$, and in which $\{\mathbf{v}, \mathbf{w}\}$ stands for the Poisson bracket of two divergence-free vector fields $\mathbf{v}$ and $\mathbf{w}$.

For a two-dimensional incompressible flow in a domain of $D \subset \mathbb{R}^2$, the right-hand side of the equation is the Poisson bracket of the *stream function* ψ and *vorticity function* $\omega = \Delta\psi$ of the vector field $\mathbf{v} = \operatorname{sgrad} \psi$:

$$\dot{\omega} = -\{\psi, \omega\}, \tag{11.2}$$

where $v_x = -\partial\psi/\partial y,\ v_y = \partial\psi/\partial x$, and $\{\psi, \omega\} = \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}$. Indeed, $-\{\mathbf{v}, \mathbf{w}\} = L_{\mathbf{v}}\mathbf{w}$ for any vector field $\mathbf{w}$. For a plane velocity field $\mathbf{v} = (v_1, v_2, 0)$, the vorticity is a function (the third component of the vorticity field $\mathbf{w} = (0, 0, \omega)$), and $L_{\mathbf{v}}\mathbf{w} = (0, 0, -L_{\mathbf{v}}\omega)$ (vector fields are transported forward, but functions are transported backward). Finally, we transform Equation (11.1) to Equation (11.2) by applying the definition of the Hamiltonian (or stream) function ψ for the field $\mathbf{v}$: $\quad -L_{\mathbf{v}}\omega = -\{\psi, \omega\}$.

The first approximation scheme we discuss for this equation goes back to Helmholtz. It replaces a smooth vorticity function ω in $D \subset \mathbb{R}^2$ by a collection of *vortices*, i.e., by a vorticity distribution supported at a finite number of points in D. Note that this scheme is also applied to a vorticity field in $\mathbb{R}^3$, where a smooth field is approximated by a singular one supported on a finite number of straight lines.

As we shall see, the motion of such isolated vortices is governed by a Hamiltonian system of ordinary differential equations. The corresponding Hamiltonian function is the logarithmic potential, i.e., a linear combination of the logarithms of the distances between the vortices (coefficients being the products of the vortex intensities).

Consider N vortices with *circulations* (i.e., the velocity circulation around the vortex) $k_i,\ i = 1, \ldots, N$, in the plane $\mathbb{R}^2$. Then the vorticity at any moment will be concentrated at N points, and the circulations at each of them will remain constant forever. Denote the (Cartesian) coordinates of the vortices in the plane by $z_i := (x_i, y_i),\ i = 1, \ldots, N$. It is convenient to write down the evolution of vortices as a dynamical system in the configuration space for the N-vortex system, the space $\mathbb{R}^{2N}$ with coordinates $(x_1, y_1, \ldots, x_N, y_N)$ and symplectic structure $\sum k_i\, dy_i \wedge dx_i$.

PROPOSITION 11.1 (SEE, E.G., [Kir]). *The vortex evolution is then given by the following system of Hamiltonian canonical equations:*

$$\begin{cases} k_i\,\dot{x}_i = \frac{\partial H}{\partial y_i}, \\ k_i\,\dot{y}_i = -\frac{\partial H}{\partial x_i}, \end{cases} \tag{11.3}$$

$1 \le i \le N$, *where the Hamiltonian function H is*

$$H = -\frac{1}{\pi}\sum_{i<j} k_i k_j \;\ln|z_i - z_j|,$$

and $|z - z_i| = \sqrt{(x - x_i)^2 + (y - y_i)^2}$.

PROOF. On the plane, the vorticity function ω describing a point vortex system has the form of a linear combination of the δ-functions:

$$\omega(z) = \sum_{i=1}^{N} k_i\;\delta(z - z_i).$$

To derive the equation of vortex evolution we first find the corresponding stream function ψ such that $\Delta\psi = \omega$. Our choice of the vorticity ω implies that the stream function is the linear combination of the fundamental solutions of the two-dimensional Laplace equation

$$\psi(z) = \frac{1}{2\pi}\sum_{i=1}^{N} k_i\;\ln|z - z_i|$$

(plus any harmonic function, which is assumed to be zero due to the vanishing boundary conditions at infinity of $\mathbb{R}^2$).

By substituting these explicit expressions for $\omega(z)$ and $\psi(z)$ into the Euler equation (11.2), we obtain that every vortex will evolve according to the following law:

$$k_j\;\dot{z}_j = \operatorname{sgrad}\big|_{z=z_j}\;\psi(z) = \frac{1}{2\pi}\sum_{i=1,\;i\neq j}^{N} k_i\;\operatorname{sgrad}\big|_{z=z_j}\;(\ln|z - z_i|).$$

The function $\psi(z)$ has a singularity at z_j, but this does not affect the motion of this vortex. This is why we can subtract the contribution of the vortex influence on itself when writing its evolution equation.

Rewritten in (x_i, y_i)-coordinates, the latter gives us the required Hamiltonian system (11.3). □

According to Helmholtz [Helm], in the case of $N = 2$, the two vortices rotate uniformly in the plane $\mathbb{R}^2$ about their common "mass center" (or rather "center of vorticity") $z = (k_1 z_1 + k_2 z_2)/(k_1 + k_2)$. In particular, if the circulations k_1 and k_2 are of the same sign, then the "mass center" is situated between the vortices, while if they are of opposite signs, then the "mass center" lies on the continuation of the line joining the vortices. If $k_1 = -k_2$, then the point vortices travel with equal velocity in parallel directions perpendicular to the line joining them.

The three-vortex problem ($N = 3$) also turns out to be integrable (unlike the classical three-body problem of gravitating mass points; see, e.g., [Poi1, Poi3]). This has already been pointed out by Kirchhoff [Kir] and illuminated in the dissertation of Gröbli [Grö], where one can find equations for evolution of the sides of the vortex triangle and explicit formulas for several special cases. An elaborate treatment of the history of the problem of three vortices can be found in [ART]. The motion of three point vortices on a sphere is considered in [KiN]. See also [Brd, BFS] for the statistical mechanics approach and [NewP] for the application of the Hannay–Berry phase (Section IV.1) to this problem.

11.B. Nonintegrability of four or more point vortices. For a general N, the Hamiltonian equations of motion (11.3) have the following four first integrals:

$$I_1 = H,\ I_2 = \sum_{i=1}^{N} k_i x_i,\ I_3 = \sum_{i=1}^{N} k_i y_i,\ I_4 = \sum_{i=1}^{N} k_i (x_i^2 + y_i^2).$$

However, these integrals are not in involution; that is, their Poisson brackets are not zero, and the system with four vortices is, generally speaking, nonintegrable [Zig1]. More precisely, the following statement holds.

Let M^5 be the (five-dimensional) manifold of all nonsingular configurations of four vortices (i.e., $z_i \neq z_j$ if $i \neq j$). This manifold is the quotient of all ordered quadruples of points in $\mathbb{R}^2$ over the three-dimensional group $E(2)$ of all motions of the plane. The quotient M^5 is a smooth manifold, since the group $E(2)$ acts on the set of ordered quadruples without fixed points.

THEOREM 11.2 ([Zig1]). *For sufficiently small $\epsilon > 0$, the dynamical system of four vortices with circulations $|k_i - 1| < \epsilon,\ i = 1, 2, 3,\ |k_4| < \epsilon$, has no analytic first integral in M^5 functionally independent of*

$$H = -\frac{1}{\pi} \sum_{i<j} k_i k_j \ \ln |z_i - z_j| \qquad \textit{and} \qquad F = \sum_{i<j} k_i k_j \ |z_i - z_j|^2.$$

REMARK 11.3. Chaotic behavior of systems with four vortices was already hinted at by Poincaré in [Poi1]. Numerical evidence of it was discussed by E. Novikov [NovE].

In spite of the fact that the 4-vortex system is generally nonintegrable, the KAM theory guarantees that for any number of vortices there is a set of positive measure in the space of initial conditions for which the motion is *quasiperiodic* [Kha]. Such vortex configurations are organized in the following way: The set of all vortices is split into several groups such that the distances between the groups are much greater than those between the vortices in the groups. In this case the vortex groups interact approximately as single vortices possessing the total circulation. The actual vortex motion is obtained as a superposition of the group motion and the independent vortex motion within the groups.

11.C. Hamiltonian vortex approximations in three dimensions. Just as the vorticity function can be approximated by a collection of point vortices, the vorticity vector field B in $\mathbb{R}^3$ can be taken to be supported on (several) curves.

Note that the corresponding closed two-form $\omega = i_B\mu$, which is the result of contraction of the field B with the volume form μ in $\mathbb{R}^3$, is assumed to be a δ-type differential form, or "current" in the sense of De Rham [DeR]. For the δ-two-form supported on a curve in $\mathbb{R}^3$, the integral over any two-dimensional surface is the algebraic number of intersections of this surface with the supporting curve.

The Euler equation (11.1) defines the evolution law for the vortex curves. Unlike the case of a two-dimensional fluid, the dynamics of such curves still constitute an infinite-dimensional system, though of "much smaller dimension" than the original equation on a smooth vorticity field. The position of every vortex curve is defined by three functions of one variable, while each component of a generic vorticity field in $\mathbb{R}^3$ is a function of three variables.

The dynamics of one smooth vortex curve in $\mathbb{R}^3$ is mathematically very interesting. The first approximation of the vortex motion, where only "local" interaction is considered, turns out to be a completely integrable system. It is known in various contexts under different names: filament equation, ferromagnetic equation, nonlinear Schrödinger equation, Landau–Lifschitz equation for the group $SO(3)$, the Betchov–Da Rios equation, etc. (see the discussion of relations between them in Section VI.3).

The inclusion of the second, already nonlocal, term into the approximation breaks the integrability (see [KlM]).

A more straightforward finite-dimensional model of the Euler equation in three dimensions is an approximation of the velocity and vorticity functions at a finite number of points; see [But, Ose2]. The Clebsch variables provide another way to deal with the canonical Hamiltonian structure in calculations. They are defined on the space that is twice as big as the space of all divergence-free vector fields (or its dual); see [M-W, Zak] and Section VI.2.

11.D. Finite-dimensional approximations of diffeomorphism groups. So far, we have been dealing with finite-dimensional models for hydrodynamical systems. However, in a number of two-dimensional cases, the entire group structure behind the fluid dynamics can, in some sense, be approximated as well.

Consider an incompressible fluid on a two-dimensional torus T^2 whose configuration space is the group $S\mathrm{Diff}\,(T^2)$ of area-preserving (or symplectic) diffeomorphisms of T^2. We show below (following [FZ, FFZ]) that this group can be "approximated" by the groups $SU(n)$ as $n \to \infty$. More precisely, the Lie algebra $S\mathrm{Vect}\,(T^2)$ admits a continuous deformation known as the family of so-called *sine*-algebras. The latter are infinite-dimensional algebras, and for integral values of the parameter, the finite-dimensional truncations of them turn out to be exactly the algebras $\mathfrak{su}(n)$. The limit of the dual spaces "respects" the Poisson brackets and the structure of Casimir functions, and has been successfully used to approximate the Euler equation in a Hamiltonian way; see [Ze1].

For a two-dimensional torus $T^2 = \{(x_1, x_2) \mod 2\pi\}$, we consider the Lie algebra $S_0\mathrm{Vect}(T^2)$ of all divergence-free vector fields on the torus with *single-valued* stream functions. The flows generated by those vector fields "do not shift" the total fluid mass. Such stream functions can be assumed to have zero mean. We complexify our Lie algebra, commutator $[\ ,\ ]$, and other operations, and then choose a basis L_k in the form of Fourier exponents $e^{i(k,x)}$, $k = (k_1, k_2) \in \mathbb{Z}^2 \setminus 0$, whose value at a point (x_1, x_2) is $\exp\ (i(x_1k_1 + x_2k_2))$.

The commutators of the basis elements L_k in the Lie algebra $S_0\mathrm{Vect}(T^2)$ are

$$[L_k, L_\ell] = (k \times \ell)\ L_{k+\ell}, \tag{11.4}$$

where $k \times \ell = k_1\ell_2 - k_2\ell_1$ is the (oriented) area of the parallelogram spanned by k and ℓ; see [Arn16] and Section IV.3.

On the other hand, the commutation relations in the algebras $\mathfrak{sl}(n, \mathbb{C})$ "approximate" those in (11.4) as $n \to \infty$ in the following sense (see [FFZ]). Fix some odd n and consider the following two matrices in $\mathfrak{sl}(n, \mathbb{C})$:

$$F = \ \mathrm{diag}\ \ (1, \varepsilon, \dots, \varepsilon^{n-1}) \qquad \text{and} \qquad H = \begin{pmatrix} 0 & 1 & & 0 & 0 \\ & \ddots & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & & \ddots & 1 \\ 1 & 0 & \dots & & 0 \end{pmatrix},$$

where ε is a primitive n^{th} root of unity and may be taken as, e.g., $\varepsilon = \exp(-4\pi i/n)$. The matrices obey the identities $HF = \varepsilon FH$ and $F^n = H^n = 1$.

Define $n^2 - 1$ matrices J_k, $k = (k_1, k_2) \in \mathbb{Z} \times \mathbb{Z} \pmod n$ and $(k_1, k_2) \neq (0, 0) \pmod n$ by setting

$$J_{(k_1,k_2)} = \varepsilon^{k_1 \cdot k_2/2} F^{k_1} H^{k_2}.$$

PROPOSITION 11.4. *The matrices J_k have zero trace and span the algebra $\mathfrak{sl}(n, \mathbb{C})$ with the following commutation relations:*

$$[J_k, J_\ell] = 2i \cdot \sin\left(\frac{2\pi(k \times \ell)}{n}\right) J_{k+\ell}. \tag{11.5}$$

PROOF is an easy calculation. Note that the set of J_k's is closed under composition and inversion:

$$J_k J_\ell = \varepsilon^{-(k \times \ell)/2} J_{k+\ell} \quad \text{and} \quad J^{-1}_{(k_1,k_2)} = J_{(-k_1,-k_2)},$$

and all these matrices have determinant equal to 1. □

As $n \to \infty$ this algebra turns into the algebra $S_0\mathrm{Vect}(T^2)$ of divergence-free vector fields on the torus, with the generators L_k and the relations (11.4) through the identification $(n/4\pi i)\ J_k \mapsto L_k$.

The Euler equation (11.2) on the torus can be approximated by making use of this limit of algebras. First, write (11.2) in terms of the Fourier components of the vorticity $\omega = \sum_m \omega_m e^{i(m,x)}$:

$$\dot{\omega}_m = \sum_k \frac{(m \times k)}{k^2}\, \omega_k\ \omega_{m-k}. \tag{11.6}$$

More generally, we recall that the Euler equation corresponding to a Lie algebra with structure constants C^k_{im} and the inertia tensor a_{ik} in coordinates $\{\ \omega_i\ \}$ has the form

$$\dot{\omega}_m = \sum_{k,l,p} a^{kp} C^l_{mp} \omega_k\ \omega_l \tag{11.7}$$

on the dual Lie algebra, where a^{kp} is the inverse inertia tensor (see Section 4). The Euler equation (11.6) for the ideal fluid on the torus is reproduced from the latter by setting

$$C^l_{mp} = (p \times m)\ \delta_{m+p-l,0} \qquad \text{and} \qquad a^{kp} = \frac{1}{k^2}\ \delta_{k+p,0}, \tag{11.8}$$

with all indices belonging to $(\mathbb{Z}\times\mathbb{Z})\setminus(0,0)$.

The $\mathfrak{sl}(n)$-approximations of the divergence-free vector fields on T^2 prompt the introduction a new dynamical system with the structure constants

$$C^l_{mp}=\frac{n}{2\pi}\sin\left(\frac{2\pi(p\times m)}{n}\right)\ \delta_{m+p-l,0},$$

with the same metric a^{kp} as in (11.8), and where all index components are now considered modulo n. By imposing a reality condition $\omega_{-m}=\bar{\omega}_m$, one obtains the approximation of the Euler hydrodynamic equation (11.6) on the torus by dynamical systems (11.7) on the algebras $\mathfrak{su}(n)$; see [Ze1].

REMARK 11.5 [Ze1]. The above limit of Lie algebras $\mathfrak{sl}(n)\to S_0\mathrm{Vect}(T^2)$ as $n\to\infty$ respects the structure of Casimir functions on the corresponding spaces. For a given n the algebra $\mathfrak{sl}(n)=\{A\in \mathrm{Mat}\,(n,\mathbb{C})\mid \mathrm{tr}\,A=0\}$ (or its real form $\mathfrak{su}(n)$) admits $n-1$ functionally independent Casimir functions, i.e., functions constant on the orbits of the (co)adjoint action:

$$\mathrm{tr}\ A^2,\ \mathrm{tr}\ A^3,\dots,\mathrm{tr}\ A^n.$$

In the limit these invariants become the momenta of the corresponding vorticity functions

$$\int_{T^2}\omega^2\ \mu,\ \int_{T^2}\omega^3\ \mu,\ \dots,\int_{T^2}\omega^n\ \mu,\dots$$

(where $\mu=d^2x$ is the standard area form on the torus), providing an infinite number of Casimirs for the area-preserving diffeomorphism group, and for the two-dimensional Euler equation. On the other hand, in three-dimensional ideal hydrodynamics there is essentially one analytic expression (helicity) for a conserved quantity of Casimir type, which makes the prospects for a reliable group approximation of the 3D fluid motion rather hopeless.

The infinite-dimensional counterpart of the integrable Euler equation of an n-dimensional rigid body (see [Man], or Section VI.1.B) was obtained in [War] by considering the limit of the algebras $\mathfrak{so}(n)$ as $n\to\infty$. For the Euler equation on the 2D sphere, an interesting model involving the rich representation theory of the dodecahedral group has been studied in [VshS].

REMARK 11.6. The algebra (11.5) is the nonextended part (also called the "cyclotomic family") of an infinite-dimensional *sine-algebra* [Hop, FFZ, FZ] with an *infinite* number of generators J_k, $k=(k_1,k_2)\in\mathbb{Z}\times\mathbb{Z}$ and $(k_1,k_2)\neq(0,0)$, and the commutation relations

$$[J_k,J_\ell]=2i\cdot\sin\left(\frac{2\pi(k\times\ell)}{\lambda}\right)J_{k+\ell}+(a\cdot k)\ \delta_{k+\ell,0}.$$

Here the constant λ is not necessarily an integer, but it is now an arbitrary complex number; $a = (a_1, a_2)$ is a fixed plane vector; the notation $\delta_{k+\ell,0}$ stands for 1 if $k = -\ell$ and 0 otherwise; and $(a \cdot k) := a_1 k_1 + a_2 k_2$. The term $(a \cdot k)\ \delta_{k+\ell,0}$ defines a nontrivial extension of the sine-algebra. We refer to [FFZ, Rog, KLR] for the definition and discussion of such extensions. Here we merely mention that in the limit $\lambda \to \infty$, after a suitable renormalization, this extension corresponds to introducing multivalued (nonperiodic) stream functions x_1 or x_2 on the torus $T^2 = \{(x_1, x_2) \bmod 2\pi\}$ whose flows are univalued periodic vector fields on T^2.

The recent interest in the sine-algebras is not only due to their hydrodynamical applications. Viewed as deformations of the Poisson algebra of functions on a two-dimensional torus, they are related to the Moyal product of functions on a linear symplectic space $\mathbb{R}^{2n}$ [Moy], the algebras of differential and pseudodifferential operators of one and several variables, the algebra of q-analogues of pseudodifferential operators [KLR], and the algebras with continuum root systems [SaV].

§12. The Navier–Stokes equation from the group viewpoint

The Euler equation of ideal hydrodynamics,

$$\dot v = -(v, \nabla)v - \nabla p \qquad (\text{or} \quad \dot\omega = -\{v, \omega\}, \quad \omega = \text{curl } v\ ),$$

is related to the Navier–Stokes equation of a viscous fluid,

$$\dot v = -(v, \nabla)v - \nabla p + f + \nu\Delta v \qquad (\text{or} \quad \dot\omega = -\{v, \omega\} + \text{curl } f + \nu\Delta\omega\ ),$$

in the same way as the classical Euler equation of a rigid body,

$$\dot m = m \times \omega,$$

is associated to a more general equation, involving friction and external angular momentum,

$$\dot m = m \times \omega + F - \nu m. \tag{12.1}$$

Here the "friction operator" ν is symmetric and positive definite. The distributed mass force f, which appeared in the Navier–Stokes equation, is similar to the external angular momentum F, and it is the origin of the fluid motion. The viscous friction $\nu\Delta v$ is analogous to the term $-\nu m$ in (12.1) slowing the rigid body motion.

The similarity becomes especially noticeable if one (following V.I. Yudovich, 1962) rewrites the equations in components in the eigenbasis of the friction operator. For example, for the Navier–Stokes equation with periodic boundary conditions one can expand the vorticity field and the force f into the ordinary Fourier series. The equations in both of the cases have the following form:

$$\dot{x}_i = \sum a_{ijk} x_j x_k + \sum f_i - \nu_i x_i. \tag{12.2}$$

In practice, one usually considers a *Galerkin approximation* in which only a finite number of terms is kept.

The first term corresponds to the Euler equation and describes the inertia motion. It follows from the properties of the Euler equation that the divergence of this term is equal to zero. Furthermore, the Euler equation of an ideal fluid in any dimension, as well as that of a rigid body, has a quadratic positive definite first integral, the kinetic energy. Therefore, for $f = \nu = 0$ the vector field on the right-hand side of Equation (12.2) is tangent to certain ellipsoids centered at the origin. This implies that during the evolution defined by this equation, at least in the finite-dimensional situation, there is neither growth nor decay of solutions (in the energy metric).

The term corresponding to the friction dominates over the constant "pumping" f when considered sufficiently far away from the origin. Hence, in that remote region, the motion is directed towards the origin, and an infinite growth of solutions is impossible (provided that the problem is finite-dimensional).

Since the "pumping" f pushes a phase point out of any neighborhood of the origin, while the friction returns it from a distance, a motion in the system of a rigid body (12.1) approaches an intermediate regime-attractor. For instance, this attractor can be a stable stagnation point or a periodic motion, while for sufficiently high dimension of the phase space it can appear to be a "chaotic" motion sensitive to the initial condition.

If the friction (or viscosity) coefficient ν is high enough, then the attractor will necessarily be a stable equilibrium position. While the parameter ν decreases (i.e., the reciprocal parameter, the Reynolds number $Re := 1/\nu$, increases), bifurcations of the equilibrium are possible, and the attractor can become a periodic motion and later a "stochastic" one.

The hypothesis that this mechanism is responsible for the phenomenon of turbulenization of a fluid motion for large Reynolds numbers has been suggested by many authors. In particular, in the Spring of 1965 A.N. Kolmogorov spelled it out at a meeting of the Moscow Mathematical Society, during a discussion of the talk by N.N. Brushlinskaya on bifurcations in Equation (12.1) [Bru]. Also in 1965, the first author, in his talk on this theory in R. Thom's seminar at IHES, formulated the conjecture that negativity of curvatures of the diffeomorphism group implies instability of fluid motion for the Euler dynamics, as well as for the corresponding attractors in the Navier–Stokes equation (see [Arn11,18]).

To normalize the attractor, A.N. Kolmogorov suggested considering the "pumping" proportional to the same small parameter ν as viscosity, and he formulated the following two conjectures for the latter case.

(1) *The weak conjecture:* The maximum of the dimensions of minimal attractors[2] in the phase space of the Navier–Stokes equations (as well as of their Galerkin approximations (12.2)) grows along with the Reynolds number $Re = 1/\nu$.
(2) *The strong conjecture:* Not only the maximum, but also the minimum of the dimensions of the minimal attractors mentioned above increases with Re.

Both of these hypotheses, with respect to two-dimensional, as well as three-dimensional hydrodynamics, still remain open.

In 1963 E. Lorenz [Lor] studied the following system in the three-dimensional phase space,

$$\begin{cases} \dot{x} = -10x + 10y, \\ \dot{y} = rx - y - xz, \\ \dot{z} = -\frac{8}{3}z + xy, \end{cases}$$

and numerically discovered an attractor with exponentially unstable motion along it for $r = 28$. This phenomenon has been called a *strange attractor*, and later it was investigated in many numerical–analytical, as well as theoretical, papers (e.g., see references in [PSS]). The above system exhibits varied and interesting properties for different r. For instance, as the parameter r decreases from 100.795 to 99.524 one observes an infinite sequence of bifurcations of period doubling of a stable periodic orbit, analogous to the successive period doublings in the Feigenbaum family of maps of a segment.

It is interesting to observe that though the Lorenz model is similar to the Galerkin approximations of the Navier–Stokes equations (12.2), there is a noticeable distinction.

For the Galerkin system (12.2) the domain where the energy grows is bounded by some ellipsoid in the phase space. Outside of that ellipsoid the energy decreases, and a phase point starts returning to the origin.

[2] An attractor is called a minimal attractor if it does not contain smaller attractors.

For the Lorenz system, the role of energy is played by a nonhomogeneous quadratic function. The instability in the Lorenz model is apparently stronger than in the Kolmogorov one. One can check how the motion along the Lorenz strange attractor sensitively depends on the initial conditions, while for the Kolmogorov model it remains a conjecture. It is proven only that a stationary flow indeed loses stability as the Reynolds number increases. The case of the sine profile $(\sin y)\,\partial/\partial x$ of the exterior force on a two-dimensional torus has been settled in [MSi]. The bifurcations in the Kolmogorov model has been studied by Yudovich, who proved the existence of a secondary regime, as well as the long-wave instability of more general steady shear flows $u(y)\,\partial/\partial x$ [Yu4].

A.N. Kolmogorov always emphasized that preservation of stability of a steady flow, even for the infinitely growing Reynolds number, would not contradict hydrodynamical experiments, under the assumption that the basin of the corresponding attractor shrinks fast enough.

The idea of a connection between the theory of hydrodynamical instability and the study of stochastization in ergodic theory of dynamical systems was repeatedly suggested by A.N. Kolmogorov for several years. For instance, in the program of his 1958/1959 seminar, which was posted on the bulletin board of the Department of Mechanics and Mathematics (Mech-Mat) at Moscow State University, he listed the following themes:

"1. Boundary value problems for hyperbolic equations whose solutions everywhere depend discontinuously on a parameter (see, for example, [Sob2]*).*

2. Problems on classical mechanics in which the eigenfunctions depend everywhere discontinuously on a parameter (a survey of these problems is contained in a lecture by Kolmogorov at the Amsterdam Congress in 1954).

3. Monogenic Borel functions and quasianalytic Gonchar functions (in the hope of applications to problems of type 1 and 2).

4. The rise of high-frequency oscillations when the coefficients of the higher derivatives tend to zero (papers of Volosov and Lykova for ordinary differential equations).

5. In the theory of partial differential equations with a small parameter at the higher derivatives, there has recently been a study of phenomena of boundary layers and interior layers converging to surfaces of discontinuity of limiting solutions, or of their derivatives, as viscosity vanishes. In real turbulence the solutions deteriorate on an everywhere dense set. The mathematical study of this phenomenon is assumed to be carried out at least on model equations (the Burgers model?).

6. Questions of stability of laminar flows. Asymptotically vanishing stability (at least on model equations).

7. Discussion of the possibility of applications to some problems in real mechanics and physics of the ideology of the metrical theory of dynamical systems.

Questions of stability of various types of spectrum. Structurally stable systems and structurally stable properties (in the latter direction, hardly anything is known for systems with several degrees of freedom!).

8. Consideration (at least on models) of the conjecture that, in the situation at the end of 5 above, in the limit the dynamical system turns into a random process (the conjecture of the practical impossibility of a long-term weather forecast)."

Constructions of the modern theory of dynamical systems, such as the Kolmogorov–Sinai entropy [Kol, Si1] measuring the degree of stochastization of a deterministic dynamics, were undertaken specifically to develop this program.

In 1970 Ruelle and Takens formulated the conjecture that turbulence is the appearance of attractors with sensitive dependence of motion on the initial conditions along them in the phase space of the Navier–Stokes equation [R-T]. In spite of the vast popularity of this paper, even the existence of such attractors still remains an open question (not to mention the earlier hypotheses of Kolmogorov on the growth in dimension of the minimal attractors).

Infinite-dimensionality of the phase space of the Navier–Stokes equation affects the foundation of the passage to the system (12.2) and to Galerkin approximations as follows. The friction operator in hydrodynamical problems is the product of viscosity ν and the Laplace operator. The absolute values of its eigenvalues ν_i increase with the order of the corresponding harmonics. Hence, the high harmonics rapidly decay for nonvanishing viscosity. It implies that a phase point of the infinite-dimensional space is attracted to the finite-dimensional one, where the coordinates are the amplitudes of the lower harmonics; see [MPa, FoT, D-O]. For a fixed viscosity the analysis of the Galerkin approximation allows one, in principle, to draw conclusions on the behavior of the actual solutions.

However, if we are interested in solution behavior as viscosity (the coefficient ν at the Laplace operator) goes to zero, then one has to consider the number of harmonics (in the Galerkin approximation) rapidly increasing as $Re = 1/\nu \to \infty$. The first explicit estimate of the Hausdorff dimension of the maximal attractor A of the Navier–Stokes equation for the case of the two-dimensional torus $\dim A \leq \text{const} \cdot \nu^{-4}$, given in [Ilsh], has been substantially improved. The best current majorant of this number is

$$\dim A \leq \frac{1}{\pi} \frac{\|f\|_{L^2} \ \cdot \ \mathrm{vol}\,(M)}{\nu^2}$$

(where f is the external force, M is a domain of finite volume, and the boundary condition is zero). It was obtained by A. Ilyin [Ily], based on [CFT]. (We refer to [B-V, Tem] for the contemporary state of the art.) As the dimension of the physical space grows, so does the number of harmonics, corresponding to the eigenvalues whose magnitude is smaller than a given number. It follows that the Galerkin approximation is to be of greater size.

The character of the first, inertia, term in (12.2) changes drastically in the passage from two-dimensional fluid flows to three-(or higher-) dimensional ones. The reason lies in the distinctions among the geometries of the coadjoint orbits of the corresponding diffeomorphism groups (or the absence of invariants of enstrophy-type for the higher-dimensional Euler equations; see Sections 9 and 11). Further, this geometry also obstructs a better foundation for the correspondence between the Galerkin approximation and the original Navier–Stokes equation in the three-dimensional case.

In the sixties most specialists in partial differential equation (with the notable exception of V.I. Yudovich) regarded the lack of global existence and uniqueness theorems for solutions of the Navier–Stokes equation as the explanation of turbulence. This point of view was never popular among physicists.

For the three-dimensional Navier–Stokes equation for small or vanishing viscosity, the existence and uniqueness theorems for an arbitrarily large period of time are still open questions.