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Polar homology and holomorphic bundles

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We describe polar homology groups for complex manifolds. The polar k-chains are subvarieties of complex dimension k with meromorphic forms on them, while the boundary operator is defined by taking the polar divisor and the Poincaré residue on it. The polar homology groups may be regarded as holomorphic analogues of the homology groups in topology. We also describe the polar homology groups for quasi-projective one-dimensional varieties (affine curves). These groups obey the Mayer–Vietoris property. A complex counterpart of the Gauss linking number of two curves in a three-fold and various gauge-theoretic aspects of the above correspondence are also discussed.

Keywords: complex manifold; divisor of poles; Poincaré residue; gauge transformations; Poisson structure

1. Introduction

In this paper we describe certain homology groups for complex projective and one-dimensional quasi-projective manifolds. These *polar homology* groups can be regarded as a complex geometric counterpart of singular homology groups in topology.

The essence of the 'polar homology' theory described below is presented in the following 'complexification dictionary':

a real manifold	$\leftrightarrow$	a complex manifold,
an orientation of the manifold	$\leftrightarrow$	a meromorphic volume form on the manifold,
manifold's boundary	$\leftrightarrow$	form's divisor of poles,
induced orientation of the boundary	$\leftrightarrow$	residue of the meromorphic form,
open manifold's infinity	$\leftrightarrow$	form's divisor of zeros,
Stokes formula	$\leftrightarrow$	Cauchy formula,
singular homology	$\leftrightarrow$	polar homology.

In short, polar k-chains in a complex projective manifold are linear combinations of k-dimensional complex submanifolds with meromorphic closed k-forms on them. The boundary operator sends such a pair (*complex submanifold, meromorphic form*) to the pair (*form's divisor of poles, form's residue at the divisor*), that is, to a $(k-1)$-chain in the same ambient manifold. The square of the boundary operator is zero, and the polar homology groups are defined as the quotients of polar cycles over polar boundaries (see § 3).

While the form's divisor of poles on a complex manifold is an analogue of the boundary of a real manifold, the form's divisor of zeros can be related to the 'infinity' of a real manifold, if the latter is non-compact (see § 4).

This parallelism between topology and algebraic geometry extends to various gauge-theoretic notions and facts. In particular, we discuss below several problems related to the correspondence of flat and holomorphic bundles. Some features of this correspondence are also present in the papers by Arnold (1971), Frenkel & Khesin (1996), Losev *et al*. (1996), Donaldson & Thomas (1998), Thomas (1997), Khesin (1997) and Khesin & Rosly (1999). Note that the gauge theory related to a version of the Chern–Simons functional on Calabi–Yau manifolds (see Witten 1995) was a motivation for the construction of these homology groups and of the relevant notion of the polar linking number (see § 5; cf. Frenkel & Todorov 2001; Khesin & Rosly 2000).

2. Polar homology of projective manifolds

We start with a heuristic motivation for polar homology and recall (following Khesin & Rosly 2000) the formal definition of the corresponding groups in the next section.

(*a*) *A holomorphic analogue of orientation*

In order to see why a meromorphic or holomorphic form on a complex manifold can be regarded as an analogue of orientation of a real manifold, we extend the analogy between de Rham and Dolbeault cochains ($\mathrm{d} \leftrightarrow \bar{\partial}$) to an analogy at the level of the corresponding *chain* complexes.

Let X be a compact complex manifold and u be a smooth $(0,k)$-form on it, $0 \leqslant k \leqslant n = \dim X$. We would like to treat such $(0,k)$-forms in the same manner as ordinary k-forms on a smooth manifold, but in the framework of complex geometry. In particular, we have to be able to integrate them over k-dimensional *complex* submanifolds in X. Recall that in the theory of differential forms, a form can be integrated over a real submanifold provided that the submanifold is endowed with an orientation. Thus we need to find a holomorphic analogue of the orientation.

For a k-dimensional submanifold $W \subset X$ equipped with a holomorphic k-form ω one can consider the following integral

$$\int_W \omega \wedge u$$

of the product of the $(k,0)$- and $(0,k)$-forms. Therefore, here we are going to regard a top degree holomorphic form ω on a complex manifold as an analogue of orientation.

(*b*) *The Cauchy–Stokes formula*

More generally, if the form ω is allowed to have first-order poles on a smooth hypersurface in W, the above integral is still well defined. The new feature brought by the presence of poles of ω manifests itself in the following relation.

Consider the integral $\int_W \omega \wedge u$, with a meromorphic k-form ω having first-order poles on a smooth hypersurface $V \subset W$. Let the smooth $(0,k)$-form u on X be $\bar\partial$-exact, that is, $u = \bar\partial v$ for some $(0,k-1)$-form v on X. Then

$$\int_W \omega \wedge \bar\partial v = 2\pi \mathrm{i} \int_V \operatorname{res}\omega \wedge v. \tag{2.1}$$

We shall exploit this straightforward generalization of the Cauchy formula as a complexified analogue of the Stokes theorem.

In the above formula, $\operatorname{res}\omega$ denotes a $(k-1)$-form on V, which is the *Poincaré residue* of ω. Namely, the form ω can be locally expressed as $\omega = \rho \wedge \mathrm{d}z/z + \varepsilon$, where $z = 0$ is a local equation of V in W and ρ (respectively, ε) is a holomorphic $(n-1)$-form (respectively, n-form). Then the restriction $\rho|_V$ is an unambiguously defined holomorphic $(n-1)$-form on V, and it is called the Poincaré residue $\operatorname{res}\omega$ of the form ω.

(*c*) *Boundary operator*

The Cauchy–Stokes formula prompts us to consider the pair (W,ω) consisting of a k-dimensional submanifold W equipped with a meromorphic form ω (with first-order poles on V) as an analogue of a compact oriented submanifold with boundary. In the polar homology theory, the pairs (W,ω) will play the role of chains, while the boundary operator will take the form $\partial(W,\omega) = 2\pi\mathrm{i}(V, \operatorname{res}\omega)$. Note that in the situation under consideration, when the polar set V of the form ω is a smooth $(k-1)$-dimensional submanifold in a smooth k-dimensional W, the induced 'orientation' on V is given by a holomorphic $(k-1)$-form $\operatorname{res}\omega$. This means that $\partial(V, \operatorname{res}\omega) = 0$, or the boundary of a boundary is zero. The latter is the source of the identity $\partial^2 = 0$, which allows one to define *polar homology* groups HP_k.

(*d*) *Pairing to smooth forms*

It is clear that the polar homology groups of a complex manifold X should have a pairing to Dolbeault cohomology groups $H^{0,k}_{\bar\partial}(X)$. Indeed, for a polar k-chain (W,ω) and any $(0,k)$-form u, such a pairing is given by the integral

$$\langle (W,\omega), u\rangle = \int_W \omega \wedge u. \tag{2.2}$$

In other words, the polar chain (W,ω) defines a current on X of degree $(n, n-k)$, where $n = \dim X$. One can see that this pairing descends to (co)homology classes by virtue of the Cauchy–Stokes formula,

$$\langle (W,\omega), \bar\partial v\rangle = \langle \partial(W,\omega), v\rangle.$$

Example 2.1. Now we are able to find out the polar homology groups HP_k of a complex projective curve Z. In this (and in any) case, all the 0-chains are cycles. Let (P,a) and (Q,b) be two 0-cycles, where P, Q are points on Z and $a, b \in \mathbb{C}$. They are polar homologically equivalent if and only if $a = b$. Indeed, $a = b$ is necessary and sufficient for the existence of a meromorphic 1-form α on Z, such that $\operatorname{div}_\infty \alpha = P + Q$ and $\operatorname{res}_P \alpha = 2\pi \mathrm{i} a$, $\operatorname{res}_Q \alpha = -2\pi \mathrm{i} b$. (The sum of all residues of a meromorphic differential on a projective curve is zero by the Cauchy theorem.) Then we can write, in terms of polar chain complex (to be formally defined in the next section), that $(P,a)-(Q,a)=\partial(Z,\alpha)$. Thus $HP_0(Z)=\mathbb{C}$.

Polar 1-cycles correspond to all possible holomorphic 1-forms on Z. On the other hand, there are no 1-boundaries, since there are no polar 2-chains in Z. Hence $HP_1(Z)\cong\mathbb{C}^g$, where g is the genus of the curve Z.

(*e*) *Polar intersections*

One can define a polar analogue of the intersection number in topology. For instance, let (X,μ) be a complex manifold equipped with a meromorphic volume form μ without zeros (its 'polar orientation'). Consider two polar cycles (A,α) and (B,β) of complimentary dimensions that intersect transversely in X (here, α and β are volume forms, or 'polar orientations', on the corresponding submanifolds). Then the polar intersection number is defined by the formula

$$\langle (A,\alpha)\cdot(B,\beta)\rangle=\sum_{P\in A\cap B}\frac{\alpha(P)\wedge\beta(P)}{\mu(P)}.$$

At every intersection point P, the ratio in the right-hand side is the 'comparison' of the orientations of the polar cycles at that point (the form $\alpha\wedge\beta$ at P) with the orientation of the ambient manifold (the form μ at P). This is a straightforward analogue of the use of mutual orientation of cycles in the definition of the topological intersection number. Note that in the polar case the intersection number does not have to be an integer. (Rather, it is a holomorphic function of the 'parameters' (A,α), (B,β) and (X,μ).)

Similarly, there is a polar analogue of the intersection product of cycles when they intersect over a manifold of positive dimension, given essentially by the same formula (see Khesin & Rosly 2000). Furthermore, one can define a polar analogue of the linking number using the same philosophy of polar chains. We discuss polar linkings, which are very close in spirit to the polar intersections, in relation to the Chern–Simons theory at the end of the paper.

Remark 2.2. Most of the above discussion extends to polar chains (A,α), where the meromorphic p-form α is not necessarily of top degree, that is, $0\leqslant p\leqslant k$, where $k=\dim_{\mathbb{C}}A$. To define the boundary operator, we have to restrict ourselves to the meromorphic forms with logarithmic singularities. The corresponding polar homology groups are enumerated by two indices k and p ($0\leqslant p\leqslant k$). One can see that the Cauchy–Stokes formula extends to this case as well, if we pair meromorphic p-forms ω on W with smooth $(k-p,p)$-forms on X.

3. Definition of polar homology groups

(*a*) *Polar chains*

In this section we deal with complex projective varieties, i.e. subvarieties of a complex projective space. By a smooth projective variety we always understand a smooth and connected one. For a smooth variety M, we denote by Ω^p_M the sheaf of holomorphic p-forms on M. The sheaf $\Omega^{\dim M}_M$ of forms of the top degree on M will sometimes be denoted by K_M.

The space of polar k-chains for a complex projective variety X, $\dim X=n$, will be defined as a $\mathbb{C}$-vector space with certain generators and relations.

Definition 3.1. The space of *polar k-chains* $\mathcal{C}_k(X)$ is a vector space over $\mathbb{C}$ defined as the quotient $\mathcal{C}_k(X) = \hat{\mathcal{C}}_k(X)/\mathcal{R}_k$, where the vector space $\hat{\mathcal{C}}_k(X)$ is freely generated by the triples (A, f, α) described in (i), (ii) and (iii) below and $\mathcal{R}_k$ is defined as relations (R1), (R2) and (R3) imposed on the triples.

(i) A is a smooth complex projective variety, $\dim A = k$.

(ii) $f : A \to X$ is a holomorphic map of projective varieties.

(iii) α is a rational k-form on A, with first-order poles on $V \subset A$, where V is a normal crossing divisor in A, i.e. $\alpha \in \Gamma(A, \Omega_A^k(V))$.

The relations are as follows.

(R1) $\lambda(A, f, \alpha) = (A, f, \lambda\alpha)$.

(R2) $\sum_i (A_i, f_i, \alpha_i) = 0$, provided that $\sum_i f_{i*}\alpha_i \equiv 0$, where $\dim f_i(A_i) = k$ for all i and the push-forwards $f_{i*}\alpha_i$ are considered on the smooth part of $\bigcup_i f_i(A_i)$.†

(R3) $(A, f, \alpha) = 0$ if $\dim f(A) < k$.

Note that, by definition, $\mathcal{C}_k(X) = 0$ for $k < 0$ and $k > \dim X$.

Remark 3.2. The relation (R2) allows us, in particular, to deal with pairs instead of triples, replacing a triple (A, f, α) by a pair $(\hat{A}, \hat{\alpha})$, where $\hat{A} = f(A) \subset X$, $\hat{\alpha}$ is defined only on the smooth part of $\hat{A}$ and $\hat{\alpha} = f_*\alpha$ there. Due to relation (R2), such a pair $(\hat{A}, \hat{\alpha})$ carries precisely the same information as (A, f, α). (The only point to worry about is that such pairs cannot be arbitrary. In fact, by the Hironaka theorem on resolution of singularities, any subvariety $\hat{A} \subset X$ can be the image of some regular A, but the form $\hat{\alpha}$ on the smooth part of $\hat{A}$ cannot be arbitrary.)

The same relation (R2) also represents additivity with respect to α, that is,

$$(A, f, \alpha_1) + (A, f, \alpha_2) = (A, f, \alpha_1 + \alpha_2).$$

Formally speaking, the right-hand side makes sense only if $\alpha_1 + \alpha_2$ is an admissible form on A, that is, if its polar divisor $\mathrm{div}_\infty(\alpha_1 + \alpha_2)$ has normal crossings. However, one can always replace A with a variety $\tilde{A}$, obtained from A by a blow-up, $\pi : \tilde{A} \to A$, in such a way that $\pi^*(\alpha_1 + \alpha_2)$ is admissible on $\tilde{A}$, i.e. $\mathrm{div}_\infty(\alpha_1 + \alpha_2)$ is already a normal crossing divisor. (This is again the Hironaka theorem.) Relation (R2) says that $(A, f, \alpha_1) + (A, f, \alpha_2) = (\tilde{A}, f \circ \pi, \pi^*(\alpha_1 + \alpha_2))$.

Definition 3.3. The *boundary operator* $\partial : \mathcal{C}_k(X) \to \mathcal{C}_{k-1}(X)$ is defined by

$$\partial(A, f, \alpha) = 2\pi\mathrm{i} \sum_i (V_i, f_i, \mathrm{res}_{V_i}\, \alpha)$$

(and by linearity), where V_i are the components of the polar divisor of α, $\mathrm{div}_\infty\, \alpha = \bigcup_i V_i$, and the maps $f_i = f|_{V_i}$ are restrictions of the map f to each component of the divisor.

Theorem 3.4 (Khesin & Rosly 2000). *The boundary operator ∂ is well defined, i.e. it is compatible with the relations (R1), (R2) and (R3). Moreover, $\partial^2 = 0$.*

† See, for example, Griffiths (1976) for the definition of the push-forward (or *trace*) map on forms.

For the proof, we refer to Khesin & Rosly (2000). Note that, having proved compatibility, the relation $\partial^2 = 0$ becomes nearly evident. Indeed, it suffices to prove it for normal crossing divisors of poles. In the latter case, the repeated residue at pairwise intersections differs by a sign according to the order in which the residues are taken. Thus the contributions to the repeated residue from different components cancel out.

Definition 3.5. For a smooth complex projective variety X, $\dim X = n$, the chain complex

$$0 \to \mathcal{C}_n(X) \xrightarrow{\partial} \mathcal{C}_{n-1}(X) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{C}_0(X) \to 0$$

is called the *polar chain complex* of X. Its homology groups, $HP_k(X)$, $k = 0, \dots, n$, are called the *polar homology groups* of X.

Remark 3.6. As we mentioned before, one can similarly define the polar homology groups $HP_{k,p}(M)$ for the case of p-forms on k-manifolds, i.e. for the forms of not necessarily top degree, $p \leqslant k$. Instead of meromorphic k-forms with poles of the first order, we have to restrict ourselves to p-forms with logarithmic singularities, keeping the definition of the boundary operator ∂ intact.

4. Polar homology for affine curves

In the preceding section we introduced polar homology of projective varieties. From the point of view of topological analogy (cf. § 1), the projective varieties play the role of compact spaces. It would be useful, of course, to also have a consistent analogue of homology of arbitrary, i.e. not necessarily compact, manifolds. It is natural to expect that this latter role is played by Zariski open subsets in projective varieties, that is, by quasi-projective varieties. This is indeed the case and the definition of polar homology can be extended to the quasi-projective case, so that the polar homology groups obey certain natural properties expected from the topological analogy. In particular, they obey the Mayer–Vietoris principle.

To simplify the exposition, we shall describe here the case of dimension one only, i.e. that of affine curves.

Let X be an affine curve and $\bar{X} \supset X$ be its projective closure. We shall define the polar chains for the quasi-projective variety X as a certain subset of polar chains for $\bar{X}$, but the result will depend only on X and not on the choice of $\bar{X}$. Let us denote by D the compactification divisor, $D = \bar{X} \setminus X$. By differentials of the third kind on a complex curve, we shall understand, as usual, meromorphic 1-forms, which may have only first-order poles.

Definition 4.1. The space $\mathcal{C}_0(X)$ is the vector space formed by complex linear combinations of points in X. It is a subspace in $\mathcal{C}_0(\bar{X})$.

The vector space $\mathcal{C}_1(X)$ is defined as the subspace in $\mathcal{C}_1(\bar{X})$ generated by the triples (A, f, α), where A is a smooth projective curve, f is a map $f : A \to \bar{X}$ and α is a differential of the third kind on A that vanishes at $f^{-1}(D) \subset A$.

Proposition 4.2. *The spaces $\mathcal{C}_k(X)$, $k = 0, 1$, form a subcomplex in the polar chain complex $(\mathcal{C}_*(\bar{X}), \partial)$, which depends only on the affine curve X and not on the choice of its compactification, the projective curve $\bar{X}$.*

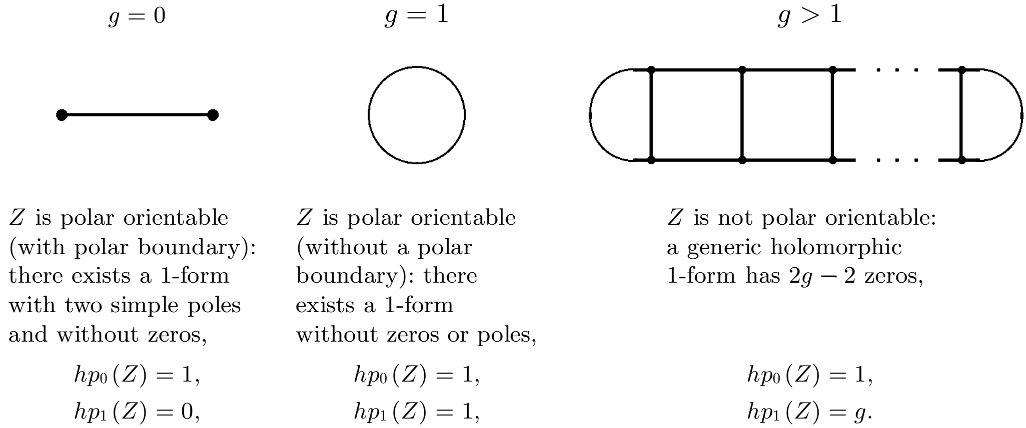

Figure 1. A smooth projective curve Z of genus g. (A rational curve is an analogue of a closed interval with two boundary points. An elliptic curve is an analogue of a circle. Higher genus curves correspond to graphs.)

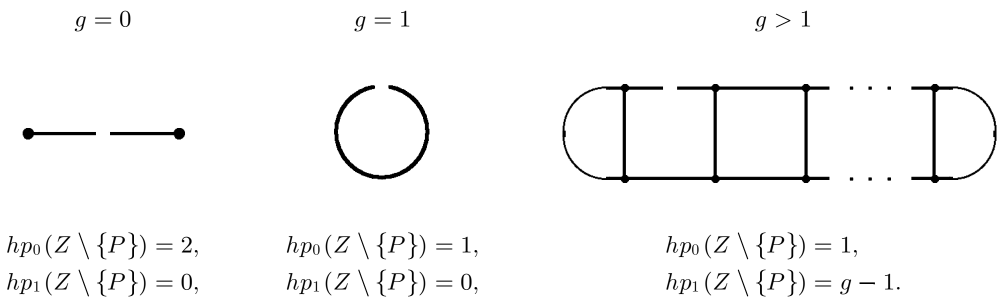

Figure 2. A smooth projective curve without a point, $Z \setminus \{P\}$.

The resulting homology groups of the chain complex $(\mathcal{C}_*(X), \partial)$ are denoted as before by $PH_k(X)$ and are called polar homology groups of X also in this case of an affine X.

Example 4.3. Let us consider a smooth projective curve of genus g without a point, $Z \setminus \{P\}$. Then, for the dimensions of polar homology groups, $hp_k(X) = \dim HP_k(X)$, we get

$$\begin{aligned} &hp_0(Z \setminus \{P\}) = 2, \quad hp_1(Z \setminus \{P\}) = 0, \qquad g = 0, \\ &hp_0(Z \setminus \{P\}) = 1, \quad hp_1(Z \setminus \{P\}) = g - 1, \quad g \geqslant 1. \end{aligned}$$

Indeed, the space $HP_1(Z \setminus \{P\})$ is the space of holomorphic 1-differentials on Z that vanish at P. To calculate $HP_0(Z \setminus \{P\})$ in the case $g \geqslant 1$, it is sufficient to notice that for any two points $Q_1, Q_2 \in Z \setminus \{P\}$, the 0-cycle $(Q_1, q_1) + (Q_2, q_2)$ is homologically equivalent to zero if and only if $q_1 + q_2 = 0$ (the same condition as in the case of a non-punctured curve, cf. example 2.1). In the case of $g = 0$, an analogous statement requires three points to be involved (unlike the case of a non-punctured projective line): the corresponding 1-form on $\mathbb{CP}^1$ has to have at least one zero, and hence at least three poles. We collect the results about the curves in figures 1 and 2 (where we

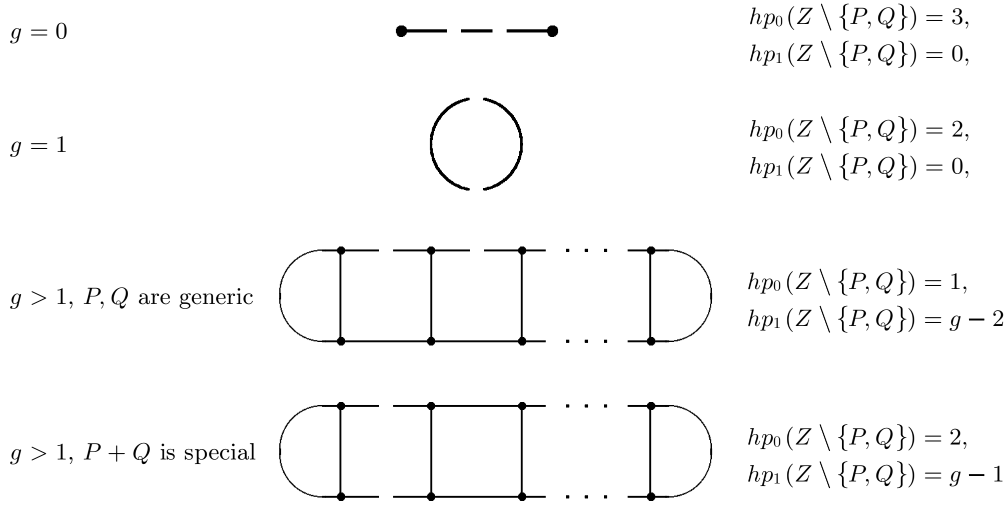

Figure 3. A smooth projective curve without two points, $Z \setminus \{P, Q\}$.

depict the complex curves by graphs, such that polar homology groups of the curves coincide with singular homology groups of the corresponding graphs).

In a similar way, for a smooth projective curve without two points, $Z \setminus \{P, Q\}$, we get the results summarized in figure 3. Here, one has to distinguish the case of generic points P and Q, and the case when $P + Q$ is a special divisor and there are more 1-differentials with zeros at P, Q than generically.

Theorem 4.4 (The Mayer–Vietoris sequence). *Let a complex curve X (either affine or projective) be the union of two Zariski open subsets U_1 and U_2, $X = U_1 \cup U_2$. Then the following Mayer–Vietoris sequence of chains is exact:*

$$0 \to \mathcal{C}_k(U_1 \cap U_2) \xrightarrow{i} \mathcal{C}_k(U_1) \oplus \mathcal{C}_k(U_2) \xrightarrow{\sigma} \mathcal{C}_k(X) \to 0.$$

Here, the map σ represents the sum of chains,

$$\sigma : a \oplus b \mapsto a + b,$$

and the map i is the embedding of the chain lying in the intersection $U_1 \cap U_2$ as a chain in each subset U_1 and U_2,

$$i : c \mapsto (c) \oplus (-c).$$

This implies the following exact Mayer–Vietoris sequence in polar homology:

$$\cdots \to HP_k(U_1 \cap U_2) \xrightarrow{i} HP_k(U_1) \oplus HP_k(U_2) \xrightarrow{\sigma} HP_k(X) \to HP_{k-1}(U_1 \cap U_2) \to \cdots.$$

The proof of this theorem in the case of curves readily follows from definitions of polar homology groups (by using a resolution if the curve is singular). One can see that such a proof essentially repeats the considerations with topological homology of appropriate one-dimensional cell complexes (i.e. graphs), as it is illustrated in examples 2.1 and 4.3, as well as in figures 1–3 above.

5. Connections and gauge transformations on complex curves and surfaces

The same philosophy of holomorphic orientation can be applied to field-theoretic notions in the following way. Suppose we have a functional

$$\mathcal{S}(\varphi) = \int_M L(\varphi, \partial_j \varphi)$$

on *smooth* fields φ (e.g. functions, connections, etc.) on a real (oriented) manifold M, and this functional is defined by an n-form L, which depends on the fields and their derivatives.

Then, on a complex n-dimensional manifold X equipped with a 'polar orientation', i.e. with a holomorphic or meromorphic n-form μ, a complex counterpart $\mathcal{S}_{\mathbb{C}}$ of the functional $\mathcal{S}$ can be defined as follows:

$$\mathcal{S}_{\mathbb{C}}(\varphi) = \int_X \mu \wedge L(\varphi, \bar{\partial}_j \varphi).$$

Here, φ stands for *smooth* fields on a complex manifold X. Now the $(0, n)$-form L is integrated against the holomorphic orientation μ over X.

Furthermore, the interrelation between the extremals of the real functional $\mathcal{S}(\varphi)$ (on smooth fields) on the real manifold M and the boundary values of those fields on ∂M (cf., for example, Schwarz 1998) is replaced by the analogous interrelation for the complex functional $\mathcal{S}_{\mathbb{C}}(\varphi)$ (still on smooth fields) on (X, μ), i.e. a complex manifold X equipped with polar orientation μ, and on its polar boundary, $\partial(X, \mu) = 2\pi \mathrm{i}(\operatorname{div}_\infty \mu, \operatorname{res} \mu)$.

Below we demonstrate some features of the above-mentioned parallelism for gauge transformations and connections on curves and surfaces (cf. Donaldson & Thomas 1998; Khesin & Rosly 1999 for other examples).

(a) *Affine and double-loop Lie algebras*

Our first example is the correspondence between the affine Kac–Moody algebras on a circle ($\mathbb{R}$ case) and the Etingof–Frenkel Lie algebras of currents over an elliptic curve ($\mathbb{C}$ case) (see Etingof & Frenkel 1994).

We will use the following notations throughout this section. Let G be a simple simply connected Lie group that is supposed to be *compact* in the $\mathbb{R}$ case and *complex* in the $\mathbb{C}$ case; $\mathfrak{g} = \operatorname{Lie}(G)$ its Lie algebra. Fix some smooth vector G-bundle $\mathcal{E}$ over a manifold M (either real or complex). The notation G^M (respectively, $\mathfrak{g}^M$) stands for the Lie group (respectively, Lie algebra) of C^∞-smooth gauge transformations of $\mathcal{E}$.

Definition 5.1.

($\mathbb{R}$) An affine Lie algebra $\hat{\mathfrak{g}}^S$ is the one-dimensional central extension of the loop algebra $\mathfrak{g}^S = C^\infty(S^1, \mathfrak{g})$ (i.e. the gauge algebra over a circle) defined by the following 2-cocycle:

$$c(U, V) = \int_{S^1} \operatorname{tr}(U \, \mathrm{d}V) \quad \text{for } U, V \in \mathfrak{g}^S.$$

(ℂ) An elliptic (or double-loop) Lie algebra $\hat{\mathfrak{g}}^E$ is a one-dimensional (complex) central extension of the gauge algebra $\mathfrak{g}^E$ over an elliptic curve E by means of the following 2-cocycle,

$$c(U,V) = \int_E \alpha \wedge \operatorname{tr}(U\bar{\partial}V),$$

where α is a holomorphic 1-form on E (its 'holomorphic orientation') and $U, V \in \mathfrak{g}^E$ (see Etingof & Frenkel 1994).

The original definition in Etingof & Frenkel (1994) was for the case of the current algebra $\mathfrak{g}^E = C^\infty(E,\mathfrak{g})$. However, it is valid in a more general case, which we need, for the group of gauge transformations of a bundle $\mathcal{E}$ not necessarily of degree zero.

The dual spaces to both affine and elliptic Lie algebras have a very natural geometric interpretation. Denote by $\mathcal{A}^M$ the infinite-dimensional affine space of all smooth connections (respectively, of all $(0,1)$-connections) in the G-bundle $\mathcal{E}$ over real (respectively, complex) manifold M.

Note that over a real curve all connections are necessarily flat. Analogously, over a complex curve every $(0,1)$-connection defines a structure of holomorphic bundle in $\mathcal{E}$ (since for such connections the curvature component $F^{0,2}$ is identically zero).

Proposition 5.2.

(ℝ) *The space $\mathcal{A}^S := \{\mathrm{d} + A \mid A \in \Omega^1(S^1,\mathfrak{g})\}$ of smooth G-connections over the circle S^1 can be regarded as (a hyperplane in) the dual space to the affine Lie algebra $\hat{\mathfrak{g}}^S$: the gauge transformations coincide with the coadjoint action. Coadjoint orbits of the affine group, or the symplectic leaves of the linear Lie–Poisson structure on the dual space $(\hat{\mathfrak{g}}^S)^*$, consist of gauge-equivalent connections and differ by (the conjugacy class of) the holonomy around S^1 (see, for example, Pressley & Segal 1986).*

(ℂ) *The space of $(0,1)$-connections $\{\bar{\partial} + A(z,\bar{z}) \mid A \in \Omega^{0,1}(E,\mathfrak{g})\}$ in the bundle $\mathcal{E}$ over the elliptic curve E can be regarded as (a hyperplane in) the dual space $(\hat{\mathfrak{g}}^E)^*$ of the elliptic Lie algebra. The symplectic leaves of the Lie–Poisson structure in the dual space $(\hat{\mathfrak{g}}^E)^*$ are enumerated by the equivalence classes of holomorphic G-bundles (or different holomorphic structures in the smooth bundle $\mathcal{E}$) over the curve E (see Etingof & Frenkel 1994).*

Remark 5.3. Feigin & Odesski (1998) found a very interesting class of Poisson algebras (as well as their deformations, associative algebras) given by certain quadratic relations, and associated to a given complex G-bundle $\mathcal{E}$ over an elliptic curve E. It turned out that the symplectic leaves of those Poisson brackets are enumerated by the isomorphism classes of holomorphic structures in $\mathcal{E}$, i.e. by the very same objects as the orbits of elliptic Lie algebras. Therefore, it would be interesting to compare the transverse Poisson structures to the orbits of double-loop Lie algebras with the transverse structures to the symplectic leaves of the Feigin–Odesski quadratic Poisson brackets.

(b) *Gauge transformations over real surfaces*

Let P be a real two-dimensional oriented manifold, possibly with boundary $\partial P = \bigcup_j \Gamma_j$. Let $\mathcal{A}^P$ be the affine space of all smooth connections in a trivial G-bundle $\mathcal{E}$ over P. It is convenient to fix any trivialization of $\mathcal{E}$ and identify $\mathcal{A}^P$ with the vector space $\Omega^1(P, \mathfrak{g})$ of smooth $\mathfrak{g}$-valued 1-forms on the surface,

$$\mathcal{A}^P = \{\mathrm{d} + A \mid A \in \Omega^1(P, \mathfrak{g})\}.$$

The space $\mathcal{A}^P$ is, in a natural way, a symplectic manifold with the symplectic structure

$$W := \int_P \mathrm{tr}(\delta A \wedge \delta A),$$

where δ is the exterior differential on $\mathcal{A}^P$ and $\wedge$ denotes the wedge product both on $\mathcal{A}^P$ and P. The symplectic structure W is invariant with respect to the gauge transformations

$$A \mapsto g^{-1}Ag + g^{-1}\mathrm{d}g,$$

where g is an element of the group of gauge transformations, G^P, i.e. it is a smooth G-valued function on the surface P. However, if the surface P has a non-empty boundary, this action is not Hamiltonian. In this case, the centrally extended group $\hat{G}^P$ of gauge transformations on the surface acts on $\mathcal{A}^P$ in a Hamiltonian way.

We are interested in the quotient of the subset of flat connections $\mathcal{A}^P_{\mathrm{fl}} \subset \mathcal{A}^P$ over the gauge groups action of $\hat{G}^P$,

$$\mathcal{M}^P_{\mathrm{fl}} = \mathcal{A}^P_{\mathrm{fl}}/\hat{G}^P = \{\mathrm{d} + A \in \mathcal{A}^P \mid \mathrm{d}A + A \wedge A = 0\}/\hat{G}^P.$$

The moduli space $\mathcal{M}^P_{\mathrm{fl}}$ is a finite-dimensional manifold (with orbifold singularities); it can also be described as the space of representations of the fundamental group $\pi_1(P)$ in G modulo conjugation.

The manifold $\mathcal{M}^P_{\mathrm{fl}}$ can be endowed with a Poisson structure. Its definition and properties can be conveniently dealt with by means of the Hamiltonian reduction $\mathcal{A}^P//\hat{G}^P$.

Theorem 5.4.

(1) *If the surface P has no boundary, then the space $\mathcal{M}^P_{\mathrm{fl}}$ of flat G-connections modulo gauge transformations on a surface P is symplectic (see Atiyah & Bott 1982).*

(2) *If $\partial P = \bigcup_j \Gamma_j$, then the moduli space $\mathcal{M}^P_{\mathrm{fl}}$ on a surface P with holes inherits a Poisson structure from the space of all (smooth) G-connections. The symplectic leaves of this structure are parametrized by the conjugacy classes of holonomies around the holes (that is, a symplectic leaf is singled out by fixing the conjugacy class of the holonomy around each hole) (see Fock & Rosly 1993, 1999).*

We note that the second part of the theorem claims that the symplectic leaves of $\mathcal{M}^P_{\mathrm{fl}}$ are labelled by the coadjoint orbits of the affine Lie algebra on a circle (or of several copies of the affine algebra, with each copy situated at a different boundary component of the surface P), since those orbits are parametrized by the conjugacy classes of holonomies around the circle.

(*c*) *Gauge transformations over complex surfaces*

In this section we present a complex counterpart of the description of the Poisson structures on moduli spaces. Let Y be a compact *complex* surface ($\dim_{\mathbb{C}} Y = 2$). Choose a polar analogue of orientation, i.e. a holomorphic or meromorphic 2-form β on Y. Let β be a meromorphic 2-form on Y, which has only first-order poles on a smooth curve X. The curve $X \subset Y$ will play the role of the boundary of the surface Y in our considerations. Moreover, assume that β has no zeros (the situation analogous to a smooth oriented compact real surface). Then X is an anticanonical divisor in Y and it has to be an elliptic curve E, or may be a number of non-intersecting elliptic curves. (Example: $Y = \mathbb{CP}^2$ with a smooth cubic as an anticanonical divisor. As a matter of fact, many Fano surfaces fall into this class.) If it happens that β has no zeros and no poles (i.e. Y is 'oriented, without boundary'), it means that we deal with either a K3 or an abelian surface.

Let $\mathcal{E}$ be a smooth vector G-bundle over Y, which can be endowed with a holomorphic structure, and $\operatorname{End}\mathcal{E}$ be the corresponding bundle of endomorphisms with the fibre $\mathfrak{g} = \operatorname{Lie}(G)$. Let $\mathcal{A}^Y$ denote the infinite-dimensional affine space of smooth $(0,1)$-connections in $\mathcal{E}$. By choosing in $\mathcal{E}$ a reference holomorphic structure $\bar\partial_0$, $\bar\partial_0^2 = 0$, the space $\mathcal{A}^Y$ can be identified with the vector space $\Omega^{(0,1)}(Y, \operatorname{End}\mathcal{E})$ of $(\operatorname{End}\mathcal{E})$-valued $(0,1)$-forms on Y, i.e.

$$\mathcal{A}^Y = \{\bar\partial_0 + A \mid A \in \Omega^{(0,1)}(Y, \operatorname{End}\mathcal{E})\}.$$

We shall often write $\bar\partial$ instead of $\bar\partial_0$, keeping in mind that this corresponds to a reference holomorphic structure in $\mathcal{E}$ when it applies to sections of $\mathcal{E}$ or associated bundles.

The space $\mathcal{A}^Y$ possesses a natural holomorphic symplectic structure

$$W_{\mathbb{C}} := \int_Y \beta \wedge \operatorname{tr}(\delta A_1 \wedge \delta A_2),$$

where β is the 'polar orientation' of Y, while the other notations are the same as above. The symplectic structure $W_{\mathbb{C}}$ is invariant with respect to the gauge transformations

$$A \mapsto g^{-1}Ag + g^{-1}\bar\partial g,$$

where g is an element of the group of gauge transformations, i.e. the group of automorphisms of the smooth bundle $\mathcal{E}$. Abusing notation, we denote this group by G^Y.

Again, we will need to centrally extend the group G^Y of gauge transformations to make the action Hamiltonian. In the momentum map, the operation of taking the curvature is replaced by the mapping

$$A \mapsto \beta \wedge F^{0,2}(A) = \beta \wedge (\bar\partial A + A \wedge A).$$

When equating the result to zero, instead of the flatness condition $F(A) = 0$, we come to the relation $F^{0,2}(A) = 0$, which singles out $(0,1)$-connections defining *holomorphic* structures in $\mathcal{E}$. Denote the space of such $\bar\partial$-connections by $\mathcal{A}^Y_{\mathrm{hol}}$. The set of isomorphism classes of holomorphic structures in $\mathcal{E}$ is represented by the quotient

$$\mathcal{A}^Y_{\mathrm{hol}}/\hat G^Y = \{\bar\partial + A \in \mathcal{A}^Y \mid \bar\partial A + A \wedge A = 0\}/\hat G^Y.$$

Analogously to the moduli space of flat connections on a real surface, we would like to study the Poisson geometry of the moduli space of holomorphic bundles over a complex surface. However, the question of existence and singularities of such a moduli space is much more subtle. Suppose the bundle $\mathcal{E}$ was chosen in such a way that there exists some version of the moduli space of holomorphic structures in $\mathcal{E}$ (e.g. (semi-)stable bundles). Denote by $\mathcal{M}^Y_{\mathrm{hol}}$ the non-singular part of that moduli space. This finite-dimensional manifold can be equipped with a holomorphic Poisson structure.

Since $\mathcal{M}^Y_{\mathrm{hol}}$ is an open dense subset in the space of isomorphism classes of holomorphic bundles,

$$\mathcal{M}^Y_{\mathrm{hol}} \subset \mathcal{A}^Y_{\mathrm{hol}}/\hat{G}^Y,$$

the Poisson structure on $\mathcal{M}^Y_{\mathrm{hol}}$ can be studied by means of the Hamiltonian reduction.

Theorem 5.5.

(1) *If Y is a K3 surface or a complex torus of dimension 2, i.e. if the 2-form β is holomorphic on Y, then the moduli space $\mathcal{M}^Y_{\mathrm{hol}}$ admits a holomorphic symplectic structure (Mukai 1984).*

(2) *If β is meromorphic, the moduli space $\mathcal{M}^Y_{\mathrm{hol}}$ of holomorphic bundles possesses a (holomorphic) Poisson structure (see A. Bondal (1995, unpublished research), Bottacin* (1995) *and Tyurin* (1987)*, where the Poisson structure is given in intrinsic terms). The symplectic leaves of this structure are parametrized by the isomorphism classes of the restrictions of bundles to the anticanonical divisor $X \subset Y$ (Khesin & Rosly 1999).*

Thus the symplectic leaves of the Poisson structure on $\mathcal{M}^Y_{\mathrm{hol}}$ are distinguished by the moduli of holomorphic bundles on elliptic curve(s) X, or, similarly, by coadjoint orbits of the corresponding elliptic algebras $\hat{\mathfrak{g}}^X$ on (the connected components of) the smooth divisor $X \subset Y$.†

The above consideration can be extended with minimal changes to the case of a non-smooth divisor X, in particular, to X consisting of several components intersecting transversally. (Example: $Y = \mathbb{CP}^2$ with $\beta = \mathrm{d}x \wedge \mathrm{d}y/xy$.) In the latter case, the corresponding degeneration of the elliptic algebra $\hat{\mathfrak{g}}^X$ can be described in terms of (several copies of) the current algebra on a punctured $\mathbb{CP}^1$.

(*d*) *Chern–Simons functionals*

First, let M be a real compact three-dimensional manifold with boundary $P = \partial M$, and $\mathcal{E}$ a trivial G-bundle over M. The *Chern–Simons functional* CS on the space of G-connections $\mathcal{A}^M$ is given by the formula

$$\mathrm{CS}(A) = \int_M \mathrm{tr}(A \wedge \mathrm{d}A + \tfrac{2}{3} A \wedge A \wedge A).$$

Extremals of this functional are flat connections on M. An action functional on the fields in three dimensions defines a symplectic structure on the space of fields in two dimensions. (It arises due to the relationship between the boundary values of the fields and the solutions to the Euler–Lagrange equations; this is essentially the Hamiltonian approach to the corresponding variational problem.) In the present

† Note that the choice of isomorphism classes of bundles on X must be subject to the condition that they arise as restrictions of bundles defined over Y.

case, as it is well known (Witten 1989), the corresponding symplectic manifold is the moduli space of flat connections $\mathcal{M}_{\rm fl}^{P}$ on $P = \partial M$.

The path integral corresponding to the Chern–Simons functional can be related with invariants of links in a three-dimensional manifold (Witten 1989) and, in the simplest case of an abelian gauge group ($G = U(1)$), reproduces the definition of the Gauss linking number.

The 'holomorphic' counterpart of the Chern–Simons functional,

$$\mathrm{CS}_{\mathbb{C}}(A) = \int_Z \gamma \wedge \mathrm{tr}(A \wedge \bar{\partial} A + \tfrac{2}{3} A \wedge A \wedge A),$$

suggested by Witten (1995), can be treated to some extent similarly. Here, $\mathrm{CS}_{\mathbb{C}}(A)$ is considered as a functional on the space of $\bar{\partial}$-connections $A \in \mathcal{A}^Z$ in a trivial G-bundle $\mathcal{E}$ over a complex 3-fold Z, where Z is equipped with a meromorphic ('polar orientation') 3-form γ without zeros, but may be with poles of the first order. In such a situation, one can apply the arguments similar to the case of the ordinary Chern–Simons theory, provided that one replaces everywhere the differential d by $\bar{\partial}$ and, instead of real boundary, one deals with the polar boundary $Y := \mathrm{div}_\infty \gamma \subset Z$. The extrema of $\mathrm{CS}_{\mathbb{C}}(A)$ are given now by integrable $\bar{\partial}$-connections ($\bar{\partial}_A^2 = 0$), that is, by holomorphic bundles over Z (which are counterparts of flat connections in three dimensions). Then, at the complex two-dimensional 'boundary', one gets the symplectic manifold $\mathcal{M}_{\rm hol}^{Y}$ of moduli of holomorphic bundles over a complex surface Y (as a counterpart of the moduli space of flat connections in two real dimensions).

The holomorphic Chern–Simons theory in the case of an abelian gauge group G on a complex simply connected 3-fold Z can be discussed even further, at the level of path integrals, without much difference with its 'real' prototype (unlike the case of an arbitrary non-abelian gauge group G, which is much more complicated and still lacks a rigorous treatment) (cf. Frenkel & Todorov 2001; Thomas 1997).

(*e*) *Polar links*

In the abelian case, the quantum holomorphic Chern–Simons theory reproduces a holomorphic analogue of the linking number. Its definition can be immediately found, again, by analogy with the ordinary one.

Let Z be a complex projective three-dimensional manifold, equipped, as above, with a meromorphic 3-form γ without zeros. Consider two smooth polar 1-cycles (C_1, α_1) and (C_2, α_2) in Z, i.e. C_1 and C_2 are smooth complex curves equipped with holomorphic 1-forms. Let us take the 1-cycles that are polar boundaries. This means, in particular, that there exists a 2-chain (S_2, β_2) such that $(C_2, \alpha_2) = \partial(S_2, \beta_2)$. Suppose that the curves C_1 and C_2 have no common points and that S_2 is a smooth surface which intersects transversely with the curve C_1. Then we define the *polar linking* number of the 1-cycles above as the polar intersection number (cf. § 2 *e*) of the 2-chain (S_2, β_2) with the 1-cycle (C_1, α_1),

$$\ell k_{\rm polar}((C_1, \alpha_1), (C_2, \alpha_2)) := \sum_{P \in C_1 \cap S_2} \frac{\alpha_1(P) \wedge \beta_2(P)}{\gamma(P)}.$$

One can show that the expression above does not depend on the choice of (S_2, β_2), and has certain invariance properties mimicking those of the topological linking number within the framework of the 'polar' approach.

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