MOSCOW MATHEMATICAL JOURNAL
Volume 3, Number 3, July–September 2003, Pages 989–1011

GEOMETRY OF HIGHER HELICITIES

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To Vladimir Igorevich Arnold on the occasion of his 65th birthday

ABSTRACT. We revisit an interpretation of higher-dimensional helicities and Hopf–Novikov invariants from the point of view of the Brownian ergodic theorem. We also survey various results related to Arnold's theorem on the asymptotic Hopf invariant on three-dimensional manifolds and recent work on linking of a vector field with a foliation, the asymptotic crossing number, short path systems, and relations with the Calabi invariant.

2000 MATH. SUBJ. CLASS. 37A15, 55Q25, 76W05.

KEY WORDS AND PHRASES. Asymptotic Hopf invariant, linking number, linking form, measured foliation, ergodic theorems.

1. INTRODUCTION

The helicity of a vector field in a three-dimensional space can be thought of as a real-valued version of the integral-valued Hopf invariant for a continuous map $S^3 \to S^2$. The latter has two well-known definitions: topological or analytic ones. Topologically, the Hopf invariant of a map $S^3 \to S^2$ is defined as the linking number in S^3 of the curves which are preimages of two arbitrary generic points in S^2. Alternatively, the Hopf invariant can be defined as the integral

$$\int_{S^3} \omega \wedge d^{-1}\omega,$$

where a 2-form ω is the pullback to S^3 of any area form on S^2, normalized by the condition that the area of S^2 equals 1. The equivalence of these two definitions is a manifestation of the Poincare duality.

The integral above can serve as a definition of an invariant for *any exact* 2-form ω on S^3 [1]. This invariant is called the *helicity* or *asymptotic Hopf invariant* of the form ω (or of the corresponding divergence-free vector field, in the presence of a volume form). It is invariant with respect to (volume-preserving) diffeomorphisms of S^3 and it is independent of the choice of a primitive $d^{-1}\omega$.

The asymptotic Hopf invariant is not necessarily an integer, if ω does not come from a map of S^3 to S^2, as well as there is no straightforward analog of the homotopy

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Received April 4, 2003.
Supported in part by the ESI, PREA, and NSERC research grants.

invariance of the Hopf invariance for map after this generalization. However, its importance is related to the fact that helicity of the vorticity field $\operatorname{curl} v$ is invariant as a divergence-free field v evolves according to the Euler equation for an ideal 3D fluid

$$\partial_t v + (v \cdot \nabla) v = -\nabla p.$$

In such a flow the field $\xi = \operatorname{curl} v$ satisfies the equation of "frozenness" in the ideal fluid: $\partial_t \xi = \{v, \xi\}$, i. e., it is transported by domain diffeomorphisms. Similarly, in the setting of magnetohydrodynamics, the magnetic field B in a perfectly conducting medium is transported by the medium flow: $\partial_t B = \{v, B\}$, which implies the preservation of its helicity.

In [1] Arnold showed that for an arbitrary exact 2-form ω its helicity integral is equal to the asymptotic linking number of the trajectories of the divergence-free vector field naturally associated with this form (in the presence of a fixed volume form on S^3).

This elegant result prompted numerous generalizations in the recent literature. In this paper we relate to each other some of these new directions. In particular, we give a Brownian ergodic description of higher helicities, inspired in part by the paper [22] on Novikov's invariants. We also survey various related results, mostly concentrating on the new developments. A review and guide to the preceding literature, as well as various tools and other details not covered in this paper, can be found in [2], [20].

One of motivations for this paper was the recent discovery of the impact of topological invariants of fluid flows on the characteristics of developed turbulence, see [6]. One can hope that a better understanding of the topological nature of ideal flows could explain the meaning and give more precise corrections to the Kolmogorov exponents for correlators of the velocity field of a turbulent flow.

The topics covered in this note include a survey of the energy estimates for vector fields in the three-dimensional case by means of helicity and asymptotic crossing number [9], the modification of the latter for a solid torus and an ergodic meaning of Calabi invariant [10], [12], as well as the short path problem [25]. For higher dimensions we discuss the linking of a vector field with a foliation with transverse invariant measure [16], [17], the mutual linking of several foliation with transverse invariant measures, as well as an interpretation of the Hopf and Novikov invariants [21], [16], [22] by means of the Brownian ergodic theorem for foliations [13].

2. What is Helicity?

Let M be a simply connected three-dimensional manifold with a volume form μ, and ξ a divergence-free vector field on M. The divergence-free property means that the Lie derivative of μ along ξ vanishes: $L_\xi \mu = 0$, or, which is the same, the substitution $\omega_\xi := i_\xi \mu$ of the field ξ to the volume form μ, is a closed 2-form: $d\omega_\xi = 0$. On a simply connected manifold M (or, even, for an M with $H_1(M) = 0$) the latter implies that ω_ξ is exact: $\omega_\xi = d\alpha$ for some 1-form α, called a potential (or, primitive) 1-form. If M has boundary, we require ξ to be tangent to the boundary. (If $H_1(M) \neq 0$ for the manifold M, we confine ourselves to null-homologous fields ξ, i. e., such fields that the 2-form ω_ξ is exact.)

In this section we always follow [1] for the definitions and results, unless otherwise is cited.

Definition 2.1. The *helicity* $\mathcal{H}(\xi)$ of a null-homologous field ξ on a three-dimensional manifold M (possibly with boundary) equipped with a volume element μ is the integral of the wedge product of the form $\omega_\xi := i_\xi \mu$ and its potential:

$$\mathcal{H}(\xi) = \int_M \omega_\xi \wedge d^{-1}\omega_\xi.$$

An immediate consequence of this definition is the following

Theorem 2.2 [1]**.** *The helicity $\mathcal{H}(\xi)$ is preserved under the action on ξ of a volume-preserving diffeomorphism of M.*

Indeed, the helicity $\mathcal{H}(\xi)$ was defined without coordinates or metric. Hence, $\mathcal{H}(\xi)$ is the same for all fields that differ only by a volume-preserving change of coordinates. In this sense, helicity is a topological invariant.

The word "helicity" was coined by K. Moffatt in [19] and it reveals the topological meaning of this characteristic of a vector field (see also [20] for a historical survey).

Example 2.3. In a simply connected domain $M \subset \mathbb{R}^3$ one can rewrite the helicity of a field ξ (tangent to the boundary of M) as follows:

$$\mathcal{H}(\xi) = \int_M (\xi,\, \mathrm{curl}^{-1}\, \xi)\, d^3x,$$

where $(\ ,\ )$ is the Euclidean inner product, and $A = \mathrm{curl}^{-1}\,\xi$ is a divergence-free vector potential of the field ξ, i. e., $\nabla \times A = \xi$. (One can easily see that the integral is independent of the choice of A defined up to an addition of a gradient ∇f.)

For instance, consider a divergence-free field ξ which is identically zero except in two narrow flux tubes whose axes are two linked closed curves C_1 and C_2. Suppose that there is no net twist within each tube or, more precisely, that the field trajectories foliate each of the tubes into pairwise unlinked circles. Then one can show (see [19]) that the helicity of such a field is given by

$$\mathcal{H}(\xi) = 2\,\mathrm{lk}(C_1,\, C_2) \cdot |\mathrm{Flux}_1| \cdot |\mathrm{Flux}_2|,$$

where Flux_i is the magnetic flux of the field in the ith tube and $\mathrm{lk}(C_1,\, C_2)$ is the linking number of C_1 and C_2.

Recall, that the (Gauss) *linking number* $\mathrm{lk}(C_1,\, C_2)$ of two oriented closed curves C_1, C_2 in $\mathbb{R}^3$ is the signed number of the intersection points of one curve with an arbitrary (oriented) surface spanning the other curve. If the curves $C_i\colon S^1 \to \mathbb{R}^3$ are parameterized by parameters in $[0,\, 2\pi]$ the linking number can be given by the following Gauss integral:

$$\mathrm{lk}(C_1,\, C_2) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{(\dot{C}_1(t_1) \times \dot{C}_2(t_2),\, C_1(t_1) - C_2(t_2))}{|C_1(t_1) - C_2(t_2)|^3}\, dt_1\, dt_2. \qquad (1)$$

V. Arnold proposed the following ergodic interpretation of helicity for any divergence-free field as the average linking number of the field's trajectories.

Let ξ be a divergence-free field on M and $\{g^t\colon M \to M\}$ its phase flow. We will associate to each pair of points in M a number that characterizes the "asymptotic

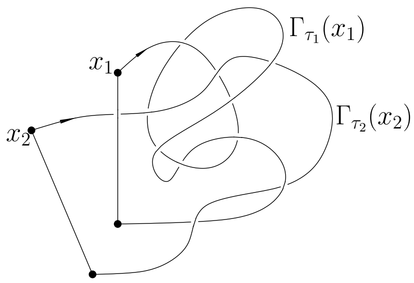

FIGURE 1. The long segments of the trajectories are closed by the "short paths"

linking" of the trajectories of the flow $\{g^t\}$ passing through these points. Given any two points x_1, x_2 in M and two large numbers T_1 and T_2, we consider "long segments" $g^t x_1$ $(0 \le t \le T_1)$ and $g^t x_2$ $(0 \le t \le T_2)$ of the trajectories of ξ issuing from x_1 and x_2. Close up these long pieces by the shortest geodesics between $g^{T_k} x_k$ and x_k. We obtain two closed curves, $\Gamma_{T_1}(x_1)$ and $\Gamma_{T_2}(x_2)$; see Fig.1. Assume that these curves do not intersect (which is true for almost all pairs x_1, x_2 and for almost all T_1, T_2). Then the linking number $\mathrm{lk}_\xi(x_1, x_2; T_1, T_2) := \mathrm{lk}(\Gamma_{T_1}(x_1), \Gamma_{T_2}(x_2))$ of the curves $\Gamma_{T_1}(x_1)$ and $\Gamma_{T_2}(x_2)$ is well-defined.

Definition 2.4. The *asymptotic linking number* of the pair of trajectories $g^t x_1$ and $g^t x_2$ $(x_1, x_2 \in M)$ of the field ξ is defined as the limit

$$\lambda_\xi(x_1, x_2) = \lim_{T_1, T_2 \to \infty} \frac{\mathrm{lk}_\xi(x_1, x_2; T_1, T_2)}{T_1 \cdot T_2},$$

where T_1 and T_2 are to vary so that $\Gamma_{T_1}(x_1)$ and $\Gamma_{T_2}(x_2)$ do not intersect.

It turns out that this limit exists (as an element of the space $L^1(M \times M)$ of the Lebesgue-integrable functions on $M \times M$) and is independent of the system of geodesics (i. e., of the Riemannian metric), see Remark 2.6 below.

Theorem 2.5 [1]**.** *Helicity of a divergence-free vector field ξ on a simply connected manifold M with a volume element μ is equal to the average pairwise linking of trajectories of this field, i. e., to the asymptotic linking number $\lambda_\xi(x_1, x_2)$ of trajectory pairs integrated over $M \times M$:*

$$\mathcal{H}(\xi) = \int_M \int_M \lambda_\xi(x_1, x_2)\, \mu_1 \mu_2.$$

Remark 2.6. In the original paper [1], instead of segments of shortest geodesics, one considered "systems of short paths" between every two points of the manifold M. These systems have to satisfy some conditions to provide the existence of $\lambda_\xi(x_1, x_2)$ almost everywhere as a pointwise limit as $T_1, T_2 \to \infty$. Such a system of "short paths" would, generally speaking, depend on a vector field. In [25] T. Vogel suggested to use the L^1-convergence, rather than the pointwise one, and showed that in the L^1-approach it is sufficient to use the system of shortest geodesics for any vector field, see Section 3.4. His approach settled in a universal way the existence question for the short paths systems. It might also shed some light on the following long-standing problem.

Problem 2.7 [1]**.** *Is the helicity of a divergence-free vector field invariant under the action of* homeomorphisms *preserving the measure on the manifold*? Here, a measure-preserving homeomorphism is supposed to transform the flow of one smooth divergence-free vector field into the flow of the other, both fields having well-defined helicities.

The above problem is a counterpart of the homotopy invariance of the classical Hopf invariant for maps $\pi\colon S^3 \to S^2$. The latter (integral-valued) invariant is also equal to the helicity of a vector field tangent to the levels of π (so that all orbits of this field are closed).

The homeomorphism invariance was proved in [10] for a field in a solid torus with nonzero longitude component. Note that for such a field there is an alternative definition of short path systems as paths that are "no longer than half the circumference," which is applicable after the action of a homeomorphism as well, see the discussion in Section 3.4. One should also mention that the class of measure-preserving homeomorphisms that are not diffeomorphisms is apparently very limited, where just few examples are known [14].

Remark 2.8. Just like the two definitions of the Hopf invariant manifest the Poincare duality, the ergodic interpretation of the helicity-type integrals in dimension three and higher can be thought of in similar terms. The "Poincare duality" here is as follows. Consider foliations of codimension k with transverse holonomy-invariant measure, and the corresponding Ruelle–Sullivan currents. (If the measure is represented by a smooth form, the kernel distribution of the form spans the foliation leaves.) The operations $\wedge$ and d^{-1} on currents (or forms) correspond, respectively, to the intersection and "filling" of the submanifolds. The final integration of a volume form over the ambient manifold corresponds to the total sum of signed intersection points of submanifolds, similar to considering the definition of linking. Thus, the invariants under consideration become properly understood average linking numbers.

3. Three-Dimensional Ramifications

In this section we consider vector fields on three-dimensional manifolds.

3.1. Bilinear and multi-linear forms. The helicity can be thought of as a quadratic form on divergence-free vector fields, as the explicit formula in Definition 2.1 shows. Similarly, one defines a symmetric form, the *cross-helicity*, for two divergence-free vector fields:

$$\mathcal{H}(\xi,\,\eta) = \int_M \omega_\xi \wedge d^{-1}\omega_\eta. \tag{2}$$

Its value turns out to be equal to the asymptotic linking of the ξ- and η-trajectories [1].

The bilinear form $H(\xi, \eta)$ corresponds to the linking of curves in 3D. For links with zero linking number there are higher invariants (Massey numbers, higher Milnor numbers, etc.). However, the adaptation of the higher link invariants to constructing the corresponding functionals on divergence-free vector fields (or on exact differential forms) requires much stronger assumptions than just vanishing of the total helicity of a field, or vanishing of the cross-helicity for a pair of such fields.

Consider, for instance, the construction of a trilinear form on divergence-free vector fields on a three-dimensional manifold [15]. Let ξ_j, $j = 1, 2, 3$, be three (null-homologous) divergence-free vector fields on a compact manifold M with a volume form μ. Consider the corresponding exact 2-forms ω_j on M, defined by $i_{\xi_j}\mu = \omega_j$. Since these forms are exact, there are potential 1-forms α_j such that $d\alpha_j = \omega_j$.

Definition 3.1. Assume, in addition, that one can choose these potential 1-forms α_j in such a way that $d(\alpha_i \wedge \alpha_j) = 0$ for all pairs $i, j = 1, 2, 3$. Then the *cross-helicity* of the three vector fields ξ_1, ξ_2, and ξ_3 is the trilinear form

$$\mathcal{H}(\xi_{1,2,3}) = \int_M \alpha_1 \wedge \alpha_2 \wedge \alpha_3.$$

One can immediately see that this form is well-defined, i. e., it is independent of the choice of the potential 1-forms α_j satisfying the condition above.

Example 3.2. An example of such a triple of fields is given by a divergence-free vector field confined to three solid tori which form a Borromean link and have no net twist inside each of the tori. One can think of such a field as a triple, where each of the three fields is supported in its own component. In this case, one can choose the potential 1-forms α_j satisfying $d(\alpha_i \wedge \alpha_j) = 0$. Note that since the pairwise linking numbers of the components are zero, so are the cross-helicities for each pair of the fields. The absence of the net twist in each solid torus implies vanishing of the corresponding helicity. The trilinear form $\mathcal{H}(\xi_{1,2,3})$ distinguishes the Borromean link from a trivial link, i. e., a set of three unlinked solitori, see [3].

Similarly, for the special class of fields confined to solid tori and without the net twist, or, a little more general, for the fields satisfying similar to the above restrictive conditions on their potentials, one can describe higher Massey numbers, provided the preceding ones vanish, see the corresponding helicity-type integrals in [4].

3.2. Energy estimates. One of the main sources of divergence-free vector fields in physics are magnetic fields. Imagine a magnetic field frozen in an infinitely conducting, but viscous medium. Then moving according to the system of the MHD equations, the medium kinetic energy will be dissipating due to the fluid viscosity, until it comes to a rest. The magnetic energy of the transported magnetic field should tend to its minimal value during the evolution, as any energy excess beyond the minimum would be converted to further motion, see [1].

It turns out that helicity of a divergence-free (magnetic) vector field is that it bounds from below field's (magnetic) energy, i. e., the square of its L^2-norm in some Riemannian metric. Namely, one has the following

Theorem 3.3 [1]**.** *For a divergence-free vector field ξ in a compact domain M*

$$\int_M (\xi,\,\xi)\,dx^3 \geq c(M)\cdot\mathcal{H}(\xi),$$

where $c(M)$ is a positive constant related to the shape and size of M.

The proof is a composition of the Schwarz inequality and the Poincare inequality, applied to the potential field $A := \mathrm{curl}^{-1}\,\xi$. (One can take the constant $c(M)$ to be reciprocal of the largest modulus of the curl^{-1} eigenvalues.) Below we discuss several directions how to sharpen this result and, in particular, to extend it to fields with zero helicity, where the above inequality is useless. For explicit estimates for the constant $c(M)$ for domains in $\mathbb{R}^3$, as well as a discussion of the geometry of extremal vector fields we refer to [5].

Remark 3.4. In a sense, the above theorem gives an estimate of a geometric quantity via topology. A geometric meaning of this inequality can be seen in the example of the field with closed orbits filling linked tori, which was discussed in Section 2. To minimize the energy of a vector field with closed orbits by acting on the field by a volume-preserving diffeomorphism, one has to shorten the length of most trajectories. (Indeed, the orbit periods are preserved under the diffeomorphism action; therefore, a reduction of the orbits' lengths shrinks the velocity vectors along the orbits.) In turn, the shortening of the trajectories implies a fattening of the solitori (since the acting diffeomorphisms are volume-preserving).

For a linked configuration the solitori prevent each other from endless fattening and therefore from further shrinking of the orbits. Therefore, heuristically, in the volume-preserving relaxation process the magnetic energy of the field supported on a pair of linked tubes is bounded from below and cannot attain too small values. In particular, a nonzero helicity (or average linking of the trajectories) of a field ξ provides a lower bound for the energy.

Note that this heuristic argument is somewhat more general than the inequality in Theorem 3.3 in the following sense. It demonstrates that there exists a lower bound for the energy for a field which has at least one linked pair of solitori as in the example above. However, the helicity of such a field might turn out to be zero, if, e. g., it has another ("mirror") pair of solitori linked in the opposite direction which makes vanish the total averaged self-linking of trajectories of the vector field. This shows that one needs a more subtle energy estimates, where, in particular, the contribution of any nontrivially linked "tube of trajectories" into the energy bound could not be canceled out.

For instance, the energy estimate via helicity is not helpful in the case of a Borromean link. Heuristically, however, there should be some lower energy bound for this field, since the components cannot be unlinked. One can show that indeed, the energy is bounded away from zero, but direct energy estimates by means of $\mathcal{H}(\xi_{1,2,3})$ are not very explicit and rather complicated, cf. [18].

Remark 3.5. More direct energy estimates were provided in [9] with the help of the notion of asymptotic crossing number. The *crossing number* of two curves is given by the Gauss integral (1) where, however, one takes the absolute value of the numerator. Geometrically, such an integral gives the number of over-crossings in a plane projection of two curves and averaged over all directions of the projection. This number is not a homotopy invariant, but one can "force" it to be invariant by minimizing over all homotopies of two given space curves.

Similarly to the linking number, one defines an asymptotic version of the crossing number for two vector fields. This way one obtains a more subtle invariant than asymptotic linking, which is, in particular, often applicable to vector fields with non-zero divergence. One of the the best energy estimates obtained with the help of this notion is as follows.

Theorem 3.6 [9]**.** *Suppose a vector field ξ in $\mathbb{R}^3$ has an invariant torus T forming a nontrivial knot K. Then*

$$\int_T (\xi,\,\xi)\,dx^3 \geq \left(\frac{16}{\pi\cdot\mathrm{Vol}(T)}\right)^{1/3}\cdot|\mathrm{Flux}\,\xi|^2\cdot(2\cdot\mathrm{genus}(K)-1),$$

where $\mathrm{Flux}\,\xi$ *is the flux of* ξ *through a crossection of* T*,* $\mathrm{Vol}(T)$ *is the volume of the solid torus, and* $\mathrm{genus}(K)$ *is the genus of the knot* K*.*

Recall, that for any knot its *genus* is the minimal number of handles of a spanning (oriented) surface for this knot. For an unknot the genus is 0, since one can take a disk as a spanning surface. For a nontrivial knot one has $\mathrm{genus}(K) \geq 1$ and, therefore, the above energy is bounded away from zero: $\int(\xi,\,\xi)\,dx^3 > 0$.

Note that there are no restrictions on the behavior of the divergence-free field inside this invariant torus, and hence this result has a wide range of applicability. In particular, it is sufficient for the field to have at least one closed knotted trajectory of the *elliptic* type. The latter means that its Poincaré map has two eigenvalues of modulus 1. Then the KAM theory implies that a generic elliptic orbit is confined to a set of nested invariant tori. Hence any such orbit ensures that the energy of the corresponding field has a non-zero lower bound.

Problem 3.7. *Does the presence of* any *nontrivially knotted closed trajectory* (of any type: hyperbolic, non-generic, etc.) *or the presence of chaotic behavior of trajectories for a field provide a positive lower bound for the energy* (even if the averaged linking of all trajectories totals zero) *and therefore prevent a relaxation of the field to arbitrarily small energies*?

Remark 3.8. The rotation field in the three-dimensional ball is an example of an opposite type: all its trajectories are *pairwise unlinked.* It was suggested by A. Sakharov and Ya. Zeldovich, and proved by M. Freedman [8], that this field can be transformed by a volume-preserving diffeomorphism to a field whose energy is arbitrarily close to 0.

3.3. Ergodic meaning of the Calabi invariant. So far we discussed vector fields with an invariant solid torus knotted in a complicated way. However, in the case of a "complicated field" in a "simple" standard solid torus the following approach based on the notion of Calabi invariant in symplectic geometry sometimes gives sharper estimates, see [10], [12]. One has to note that the asymptotic Hopf invariant (unlike the Hopf invariant for a map) is difficult to find explicitly for a somewhat generic divergence-free vector field. Here, however, we discuss the case where these calculations can be done explicitly for a field in a solid torus, coming from a generic area-preserving diffeomorphism of a two-disk.

Let D^2 be a two-dimensional disk equipped with an area-form ω. Take any primitive 1-form α, such that $\omega = d\alpha$. Consider a smooth area-preserving diffeomorphism ϕ of the disk, which is identity near the disk boundary. Then the 1-form $\phi^*\alpha - \alpha$ is closed and vanishes near ∂D. Hence there is a unique function $h(\phi)\colon D^2 \to \mathbb{R}$, which vanishes near the boundary ∂D and satisfies

$$dh(\phi) = \phi^*\alpha - \alpha.$$

Definition 3.9. The *Calabi invariant* of the area-preserving diffeomorphism ϕ is defined by the integral:

$$\mathcal{C}(\phi) := \int_{D^2} h(\phi)\,\omega.$$

One can check that this integral does not depend on the choice of the primitive 1-form α, see, e. g., [10] or [2].

Remark 3.10. Another way to define this invariant is to consider a Hamiltonian isotopy $\{\phi_t\}$, $t \in [0,\, 1]$, which starts with the identity map at $t = 0$, and at $t = 1$ coincides with ϕ. Such an isotopy can be defined by a t-dependent Hamiltonian function $H_t\colon D^2 \to \mathbb{R}$, which vanishes near the boundary and satisfies the identity

$$dH_t(\,.\,) = \omega\left(\frac{\partial \phi_t}{\partial t},\ .\right)$$

for all $t \in [0,\, 1]$. Then one can show (see, e. g., [2]) that the Calabi invariant is (twice) the volume under the graph of the Hamiltonian function H_t over $D^2 \times [0,\, 1]$:

$$\mathcal{C}(\phi) = 2\int_0^1 \int_{D^2} H_t(x)\,\omega\,dt,$$

Remark 3.11. One more definition of the Calabi invariant, due to A. Fathi, has an ergodic meaning of an average orbit braiding, and was used in [10], [12] for a variety of applications. Let $\{\phi_t\}$, $t \in [0,\, 1]$, be a Hamiltonian isotopy. Consider the map

$$\mathrm{Ang}_\phi\colon (D^2 \times D^2) \setminus \Delta \to \mathbb{R}^2,$$

which associates to any pair of points $x \neq y$ in D^2 the angular variation of the vector from $\phi_t(x)$ to $\phi_t(y)$ when t goes from 0 to 1 (here Δ stands for the diagonal). Note that this map does not depend on the isotopy. Then the Calabi invariant is the integral of this function (see [10]):

$$\mathcal{C}(\phi) = \iint_{D^2 \times D^2} \mathrm{Ang}_\phi(x,\, y)\,\omega_x \omega_y.$$

Note that the latter definition is pure two-dimensional, while the former ones allow generalizations to higher dimensional symplectic manifolds.

Problem 3.12 [10]**.** *Find a higher-dimensional analog of the angular definition of the Calabi invariant.*

Consider a divergence-free vector field ξ in a solid torus in $\mathbb{R}^3$ with a nonvanishing longitude component $\partial/\partial\theta$. It defines a Poincare map of a section, the two-dimensional disk, from $\theta = 0$ to $\theta = 2\pi$. This Poincare map is area-preserving (i. e. symplectic) for the area form $\omega = \omega_\xi$ defined by the substitution of this vector field to the volume 3-form (more precisely, by the restriction of this substitution to a cross-section of the solid torus). One can describe the asymptotic linking number of the field trajectories in the solitorus in terms of the Calabi invariant of the Poincare map. The the latter definition was used to prove the homeomorphism invariance of the asymptotic linking (i. e., helicity) of a field in a solid torus [10] and to give more precise energy estimates for such fields in a torus in terms of average braiding of their orbits [12]. The paper [11] shows, in particular, that signatures of links behave much more regularly than their linking numbers under iterations of the Poincare map. We also refer to [24] for a similar in spirit description of an asymptotic version of the Bennequin invariant for a vector field in a contact manifold and an ergodic interpretation of the Godbillon-Vey class of a foliation of codimension 1.

3.4. Short paths systems. In order to discuss linkings of infinite trajectories one needs to pass to linkings of non-closed curves. A way to do this is to introduce a "linking form."

Definition 3.13. A *linking form* on an n-dimensional compact manifold M is a $(k,\, l)$-form $LF \in \Omega^*(M \times M)$ satisfying the condition: the linking number of any two non-intersecting closed oriented compact submanifolds P and Q of linking dimensions k and l (with $k + l = n - 1$) is given by the double integral

$$\iint_{P\times Q\subset M\times M} LF.$$

For instance, the Gauss linking form in $\mathbb{R}^3$ is a linking $(1,\, 1)$-form, whose value at a point $(x,\, y) \in \mathbb{R}^3 \times \mathbb{R}^3$ on a pair of vectors v and w is

$$LF(x,\, y)|_{(v,w)} = \frac{1}{4\pi}\frac{(v\times w,\, x-y)}{|x-y|^3},$$

cf. the Gauss formula (1). One can show that a linking form exists on a manifold M provided that $b_k(M) = b_{k+1}(M) = 0$, see details in [2]. The linking form LF can be chosen to have a singularity $1/r^{n-1}$, where r is the distance to the diagonal in $M \times M$ (similar to the Gauss LF, see formula (1) for $n = 3$), i. e. to be integrable: $LF \in L^1(M \times M)$, since the diagonal is of codimension n in $M \times M$.

The introduction of such a form allows one to consider "linking of non-closed submanifolds." In particular, we define the linking of non-closed segments γ_1 and γ_2 in $\mathbb{R}^3$ by

$$\mathrm{lk}(\gamma_1,\, \gamma_2) := \iint_{\gamma_1\times\gamma_2} LF. \tag{3}$$

Recall that the linking of trajectories of a vector field ξ issuing from the points x and y is the limit

$$\lambda_\xi(x_1,\, x_2) = \lim_{T_1,T_2\to\infty}\frac{1}{T_1\cdot T_2}\,\mathrm{lk}(\Gamma_{T_1}(x_1),\, \Gamma_{T_2}(x_2)).$$

Each curve $\Gamma_{T_j}(x_j) = g_{x_j}^{[0,T_j]} \cup \Delta_{x_j}$ consists of a long piece of a trajectory $g_{x_j}^{[0,T_j]}$ and a short closing segment Δ_{x_j}. The systems of short paths (SSP) $\{\Delta\}$ should satisfy the following conditions ensuring the limit existence, see [1]. Due to the additivity of the linking number (as the double integral (3) of the linking form), we have:

$$\begin{aligned}\mathrm{lk}(\Gamma_{T_1}(x_1),\, \Gamma_{T_2}(x_2)) &= \mathrm{lk}(g_{x_1}^{[0,T_1]} \cup \Delta_{x_1},\, g_{x_2}^{[0,T_2]} \cup \Delta_{x_2})\\ &= \mathrm{lk}(g_{x_1}^{[0,T_1]},\ g_{x_2}^{[0,T_2]}) + \mathrm{lk}(g_{x_1}^{[0,T_1]},\ \Delta_{x_2}) + \mathrm{lk}(\Delta_{x_1},\ g_{x_2}^{[0,T_2]}) + \mathrm{lk}(\Delta_{x_1},\ \Delta_{x_2}).\end{aligned}$$

The first term describes the average linking (after the division by $T_1 \cdot T_2$) of the corresponding trajectories. Therefore, one has to ensure that the contribution of any of the remaining three summands is negligible. For instance, one could require that each of them does not grow faster than a linear function of T_1 or T_2 for almost all points $x_1,\, x_2 \in M$, and hence, after taking the time average (i. e., after dividing by $T_1 \cdot T_2$) their contributions to $\mathrm{lk}(\Gamma_{T_1}(x_1),\, \Gamma_{T_2}(x_2))$ tend to 0 as T_1 and T_2 go to infinity.

Remark 3.14. In [1] it was discussed that there are plenty of SSP which provide this bound *for almost all pairs* x_1 and x_2. Note that there might be some exceptional values of x_1 and x_2, where, e. g., the ξ-trajectory through x_1 winds exponentially fast on a short path for x_2. Then, of course, $\mathrm{lk}(g_{x_1}^{[0,T_1]},\, \Delta_{x_2})$ grows also exponentially, as $T_1 \to \infty$, even after the division by T_1. One can see that a small perturbation of such a system $\{\Delta\}$ destroys this exponential winding. This way, however, the definition of the SSP might depend on the field ξ. T. Vogel in [25] suggested to replace the convergence almost everywhere by the L^1 convergence, which allowed one to present some universal SSPs.

Definition 3.15 [25]**.** A system $\{\Delta\}$ is a *system of short paths* on M if for any pair of points $x_1,\, x_2 \in M$ there is a piecewise differentiable path $\Delta_{x_1,x_2} \in \{\Delta\}$ such that the limits

$$\lim_{T_1,T_2\to\infty} \frac{1}{T_1 \cdot T_2} \iint_{g_{x_1}^{[0,T_1]} \times \Delta_{x_2}} |LF|$$

for $g_{x_1}^{[0,T_1]} \times \Delta_{x_2}$, as well as those for $\Delta_{x_1} \times g_{x_2}^{[0,T_2]}$ and $\Delta_{x_1} \times \Delta_{x_2}$, vanish in the L^1 sense.

Theorem 3.16 [25]**.** *On any compact Riemannian manifold M a system of shortest geodesics joining any pair of points x_1 and x_2 is a system of short paths for any vector field ξ.*

Note that the description of the SSP might be related to the problem of the helicity invariance with respect to volume-preserving homeomorphisms. While for a field in a solid torus (the only case where this invariance was proved [10]) there is a natural system of short paths, in general one has to be sure that a SSP is not spoiled too much by a homeomorphism, in order to apply the homeomorphism invariance of the linking number for closed curves.

4. Linking of a Vector Field with a Measured Foliation

4.1. Linking of a curve with a foliation. An average linking of a measured foliation of codimension 2 and a divergence-free vector field is a direct generalization of the cross-helicity, cf. [2], [16], [17]. The corresponding helicity invariants in higher dimensions are similar to those appearing in higher-dimensional ideal hydrodynamics, an odd-dimensional ideal incompressible fluid moving according to the Euler equations. It should be mentioned however, that they do not seem to be related to higher-dimensional physical problems, similar, e. g. to the helicity-energy interaction in the MHD theory in two and three dimensions.

Let M be a closed manifold with $b_1(M) = 0$ and equipped with a volume form μ. Consider an oriented codimension 2 foliation $\mathcal{F}$ with a holonomy-invariant transverse measure β. To such a foliation one can associate a current $C(\mathcal{F})\colon \Omega^{n-2}(M) \to \mathbb{R}$, which sends any smooth $(n-2)$-form θ on M to its integral against β:

$$C(\mathcal{F})\colon \theta \mapsto \int_M \theta \wedge \beta.$$

(The integral here can be defined via a partition of unity subordinated to a foliation chart for $\mathcal{F}$: one integrates the summands over the leaf plaques in the charts, and then integrates the result over transversals using the measure β.) This current is closed and is called the *Ruelle–Sullivan cycle* of the invariant measure β [23]. Assume that this cycle is null-homologous: $[C(\mathcal{F})] = 0 \in H_{n-2}(M, \mathbb{R})$. (Geometrically one might think of this condition that the "leaves would be boundaries if they were compact manifolds," cf. Remark 2.8.)

If we imagine that the measure β is represented by a smooth 2-form on M, then necessarily this form is exact, since $[C(\mathcal{F})] = 0$. Moreover, the foliation $\mathcal{F}$ is spanned by the distribution of kernels of this form: $\beta = i_{\mathcal{F}}\mu$. In the sequel we will be often dealing with the measure β as it were a smooth 2-form, while one has to use the language of the Ruelle–Sullivan cycles to furnish the details, see the careful study in [17].

Definition 4.1. The *average linking* of a curve Γ with the measured foliation $\mathcal{F}$ is the β-measure of an arbitrary surface $\partial^{-1}\Gamma$ bounded by Γ. In other words, it is the flux of the 2-form $\beta = i_{\mathcal{F}}\ \mu$ through $\partial^{-1}\Gamma$:

$$\mathrm{lk}(\Gamma, \mathcal{F}) = \int_{\partial^{-1}\Gamma} \beta = \int_\Gamma d^{-1}\beta.$$

The motivation of this definition is as follows. Suppose, that we are given a measured foliation $\mathcal{F}$, where the total weight is supported on a single (null-homologous) compact leaf N of codimension 2. Then $\int_\Gamma d^{-1}\beta$ is equal to the linking number of N and the curve Γ. Similarly, for a foliation with compact nonsingular fibers, the number $\mathrm{lk}(\Gamma, \mathcal{F})$ gives the average for the linking numbers of the curve Γ with every fiber, cf. [16], [2].

4.2. Linking of a vector field with a foliation. By analogy with the three-dimensional case, we can now define an asymptotic and average linking numbers for a vector field and a measured foliation. For a field ξ on M denote by $\Gamma_T(x)$ a closed curve consisting of the long segment (for time $0 \le t \le T$) of the ξ-trajectory $g_\xi^t x$ starting at $x \in M$ and of a short closing path (e. g., geodesic).

Definition 4.2. An *asymptotic linking* $\mathrm{lk}_\xi(x, \mathcal{F})$ of the trajectory of a vector field ξ emanating from a point $x \in M$ with the measured foliation $\mathcal{F}$ is the time-average of the linking number of the curve $\Gamma_T(x)$ with $\mathcal{F}$:

$$\mathrm{lk}_\xi(x, \mathcal{F}) = \lim_{T\to\infty} \frac{1}{T}\,\mathrm{lk}(\Gamma_T(x), \mathcal{F}).$$

The *average linking number* of the vector field ξ and the measured foliation $\mathcal{F}$ on the manifold M equipped with the volume form μ is

$$\mathrm{lk}_\xi(\mathcal{F}) = \int_M \mathrm{lk}_\xi(x, \mathcal{F})\mu.$$

Definition 4.3. The *cross-helicity* of a vector field ξ and a codimension 2 foliation $\mathcal{F}$ with transverse holonomy-invariant measure β is

$$\mathcal{H}(\xi, \mathcal{F}) := \int_M \beta \wedge d^{-1}(i_\xi \mu),$$

cf. the 3D case (2).

Theorem 4.4 [16], [17]**.** *Let ξ be a divergence-free vector field on M with a volume form μ, and $\mathcal{F}$ a foliation of codimension 2 with a transverse holonomy-invariant measure β. Then the average linking number of the vector field ξ with the foliation $\mathcal{F}$ is equal to the cross-helicity of ξ and $\mathcal{F}$:*

$$\mathrm{lk}_\xi(\mathcal{F}) = \mathcal{H}(\xi, \mathcal{F}).$$

Proof.

$$\begin{aligned}\mathrm{lk}_\xi(\mathcal{F}) &= \int_M \mathrm{lk}_\xi(x, \mathcal{F})\,\mu = \int_M \left[\lim_{T\to\infty}\frac{1}{T}\,\mathrm{lk}(\Gamma_T(x), \mathcal{F})\right]\mu \\ &= \int_M \left[\lim_{T\to\infty}\frac{1}{T}\,\mathrm{lk}(g_x^{[0,T]}, \mathcal{F})\right]\mu = \int_M \left[\lim_{T\to\infty}\frac{1}{T}\left(\int_{g_x^{[0,T]}} d^{-1}\beta\right)\right]\mu \\ &\qquad\qquad = \int_M \left[\lim_{T\to\infty}\frac{1}{T}\int_0^T i_\xi(d^{-1}\beta)dt\right]\mu.\end{aligned}$$

The latter integrand is the time average of the function $i_\xi(d^{-1}\beta)$. By the Birkhoff ergodic theorem applied to the μ-preserving flow of the field ξ, the time average is equal to the space average of the same function:

$$\int_M i_\xi(d^{-1}\beta)\,\mu = \int_M d^{-1}\beta \wedge i_\xi\mu = \int_M \beta \wedge d^{-1}(i_\xi\mu) = \mathcal{H}(\xi, \mathcal{F}).$$

On the way, we replaced the integral over a closed curve $\Gamma_T(x)$ by the integral over a non-closed piece of trajectory $g_x^{[0,T]}$. Indeed, the difference of the two integrals is the integral of an L^1 1-form $d^{-1}\beta$ over the short path (a shortest geodesic) from x to the other endpoint of $g_x^{[0,T]}$. It is uniformly bounded by the product of the diameter of M and the norm of $d^{-1}\beta$ (note that d^{-1} is a compact operator). Hence it can be neglected after division by T and considering the limit $T \to \infty$. $\square$

Remark 4.5. Note that the proof of Arnold's theorem 2.5 is achieved similarly by applying the Birkhoff ergodic theorem to the field $\xi \times \xi$ on $M \times M$, and using the short path properties (Section 3.4). For more details on the case of a general transverse holonomy-invariant measure we refer to [17].

4.3. Linking of several measured foliations. In the next section we discuss the problem of generalizing the ergodic linking to foliations. However, there is one case where such a generalization follows immediately from the definitions above.

Consider an n-dimensional closed manifold M with a volume form μ, and several foliations of arbitrary dimensions with holonomy invariant measures on M. We assume that (at least) one of the foliations is of codimension 2, and that the sum of all codimensions of these foliations is $n+1$. Finally, assume that the foliations are null-homologous, i. e., so are the corresponding Ruelle–Sullivan cycles.

For simplicity we suppose that these foliations are given by smooth exact forms $\beta_1, \ldots, \beta_k$ as above (though the statement below can be understood in the more general context). For instance, one can take k exact 2-forms of rank 2 on an odd-dimensional manifold M^{2k-1}. Suppose that β_1 is a 2-form and defines a foliation of codimension 2.

Define the vector field ξ as the intersection of all the foliations but the first one, "normalized by" the volume form μ: $i_\xi \mu = \beta_2 \wedge \cdots \wedge \beta_k$.

Theorem 4.6. *The cross-helicity type integral $\int_M d^{-1}\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_k$ is equal to the average linking of the foliation $\mathcal{F}_1$, defined by β_1, and the vector field ξ, defined by the intersection of the others.*

In other words, the Hopf type integral above is equal to the average linking of the first foliation with the intersection of the rest, provided that the whole set has linking dimensions. The same holds for the foliations with holonomy invariant transversal measures: the intersection foliation (with singularities) has a holonomy invariant product measure. (The latter automatically vanishes at the points of non-transversality of the foliations.)

This theorem is just a specification of the one discussed in the preceding section. Note that at the points where the foliations $\mathcal{F}_2, \ldots, \mathcal{F}_k$ are not transversal, the field ξ automatically vanishes, since so does the wedge product $\beta_2 \wedge \cdots \wedge \beta_k$. The latter theorem is related to a notion of mutual linking of several submanifolds.

Definition 4.7. Let $P_1, \ldots, P_k$ be oriented closed submanifolds in $\mathbb{R}^n$ or S^n satisfying the following two conditions: a) their total mutual intersection is empty and b) the sum of their codimensions is equal to $n+1$. Then their *mutual linking number* is the signed number of the intersection points of a manifold $\partial^{-1}P_i$ spanning one of these submanifolds P_i with the intersection of all the other submanifolds.

If these submanifolds are equipped with some transversal orientations, then so are all the manifolds bounded by them and all their intersections. Hence the signs of the intersection points are well-defined. For example, it is possible to link three circles in the plane or two spheres and one circle in 3D, see [2]. These higher linking numbers naturally arise in the consideration of higher-dimensional analogs of the Chern–Simons functional [7].

Remark 4.8. The linking number of two submanifolds is symmetric or anti-symmetric according to whether the product of their codimensions is even or odd. The mutual linking number has a similar symmetry property.

The theorem above gives an ergodic interpretation of the Hopf-type integrals appearing in higher-dimensional ideal fluid dynamics: $I(\beta_1, \ldots, \beta_k) = \int_M d^{-1}\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_k$ for k closed 2-forms β_i on an odd-dimensional manifold M^{2k-1}. Note that this integral is symmetric under the permutations of β_i and does not depend on the choice of the primitive $d^{-1}\beta_1$.

5. Linking of Foliations and Brownian Motion

An attempt to generalize the ergodic view on helicity to higher dimensions encounters the following difficulty. Consider, for instance, the Hopf invariant for a map $\rho\colon S^{2k-1} \to S^k$. Just like in the $S^3 \to S^2$-case, it is defined as the linking number of the preimages in S^{2k-1} of two generic points in the target sphere S^k. Alternatively, one can define it as the integral

$$\int_{S^{2k-1}} \beta \wedge d^{-1}\beta, \tag{4}$$

where a k-form β is the pullback to S^{2k-1} of any volume form on S^k, normalized by the condition that the total volume of S^k equals 1. The analytic definition above (*generalized helicity*) extends easily to the case of *any exact* k-form β on S^{2k-1}, or to *generalized cross-helicity*

$$\int_M \beta \wedge d^{-1}\gamma \tag{5}$$

for a pair of two exact forms β and γ on an n-dimensional M, where $\deg\beta + \deg\gamma = n+1$. The problem is to give an ergodic interpretation of these integrals in terms of certain foliations associated to these forms. It is probably more natural to formulate the problem in the following converse way. Given two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of linking dimensions in M with transverse holonomy invariant measures β_1 and β_2, describe the average linking of their fibers in terms of these measures. The following definition makes the problem nearly tautological (though it becomes more interesting when we later fix a volume form on M.)

Definition 5.1. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two foliations of linking dimensions k and l with transverse holonomy-invariant measures β_1 and β_2 in a compact manifold M. The *cross-helicity* of these foliations is

$$\mathcal{H}(\mathcal{F}_1, \mathcal{F}_2) := \iint_{M_x \times M_y} LF(x, y) \wedge (\beta_1(x) \otimes \beta_2(y)) = \iint_{C_x(\mathcal{F}_1)\times C_y(\mathcal{F}_2)} LF(x, y),$$

where $C(\mathcal{F}_i)$ is the Ruelle–Sullivan cycle of the foliation $\mathcal{F}_i$, and LF is a linking form on $M \times M$.

Remark 5.2. In the case of smooth β_i the cross-helicity can be also defined as follows:

$$\mathcal{H}(\mathcal{F}_1, \mathcal{F}_2) := \int_M \beta_1 \wedge d^{-1}\beta_2 = \int_{C(\mathcal{F}_1)} d^{-1}\beta_2.$$

The equivalence of these definitions follows from the main property of the linking form: it defines an operator $\widetilde{LF}\colon \Omega^*(M) \to \Omega^*(M)$, such that for an exact k-form u the image is one of possible primitives: $LF(u) = d^{-1}u$ modulo an exact $(k-1)$-form.

Actually, this very property (or the equivalence of the above definitions) is a higher-dimensional (somewhat tautological) analog of the helicity ergodic interpretation in 3D: the linking characteristic of the foliations is expressed in the analytic form. To give a more precise meaning to the linking of two foliations we will fix a volume form on M. The latter allows one to identify differential forms (or, invariant measures for the foliations) with polyvector fields (respectively, generalized polyvector fields spanning these foliations).

Remark 5.3. Note that in dimension 3 for linking of two vector fields, as well as for linking of one field and a foliation in higher dimensions, one has a natural "time" along the field trajectories, which allows one to apply the Birkhoff ergodic theorem, as well a short path system. In higher dimension, where the interpretation should be related to linking of possibly non-compact leaves of the foliations, one is lacking both a natural expanding system of sets, comprising the linking of different leaves, and a natural generalization of short path systems.

A way around these difficulties is an application of a Brownian version of the Birkhoff ergodic theorem, cf. [22]. The latter allows one to consider averaging of a function over leaves of a foliation. This averaging is made with the help of a leaf Laplace–Beltrami operator, and can be thought of as the large time limit of averaging over Brownian trajectories along the leaves. First we recall several notions from foliation theory, see [13].

Definition 5.4. Let $(M, \mathcal{F})$ be a compact foliated Riemannian manifold. Consider the leaf diffusion operators $D(t)$ for the leaf Laplacian Δ_F. Then the *Brownian average* of a function f on M is the following function

$$\tilde{f}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} (D(t)f)(x).$$

Here the leaf diffusion operator $D(t)$ is defined as the limit $R \to \infty$ of the truncated diffusion operator:

$$D(t,\, R)f(x) = \int f(z)p(x,\, z,\, t,\, R)\, dz,$$

where $p(x, z, t, R)$ equals the heat diffusion kernel $p(x, z, t)$ if the leaf distance between x and z is not greater than R and it equals zero otherwise. (The diffusion operators $D(t, R)$ can be thought of as averaging a given heat source after the time R, while the operator $f \to \tilde{f}$ as averaging f over Brownian paths corresponding to the leaf Laplace–Beltrami operator, Fig. 2.)

Although for foliations with transverse holonomy invariant measure there are various choices for a convenient measure in the ambient manifold, we will employ the following general existence result due to L. Garnett.

Theorem 5.5 [13]**.** 1) *Any compact foliated Riemannian manifold* $(M, \mathcal{F})$ *has a nontrivial harmonic measure* ν*, i. e., probability measure such that* $\langle \Delta_F f,\, \nu\rangle = 0$ *for all* f *smooth in the leaf direction, where* Δ_F *is the leaf Laplacian*;

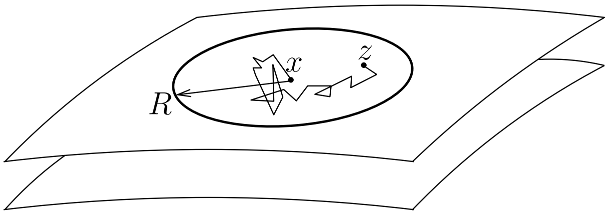

FIGURE 2. Brownian trajectories on leaves

2) *Let ν be a finite harmonic measure. Then the ergodic theorem holds, using the leaf diffusion operators $D(t)$ as follows: For any ν-integrable function f there exists an ν-integrable Brownian average $\tilde{f}$ which is constant along the leaves and*

$$\langle \tilde{f}, \nu \rangle = \langle f, \nu \rangle.$$

This theorem allows one to interpret ergodically the above integrals (5) by considering a Brownian motion along the leaves of the foliations and evaluating a certain linking form along this motion. As we will see, the corresponding number measuring the mutual linking of the leaves of the two foliations coincides with the Hopf-type integral.

Define a Brownian average linking of two foliation leaves as follows. Let $(M, \mathcal{F}_1, \mathcal{F}_2)$ be a compact Riemannian manifold with two foliations of linking dimensions k and l. Given a volume-form μ on M, these foliations $\mathcal{F}_j$ can be defined by (generalized) polyvector fields (which we denote by the same letters), such that $i_{\mathcal{F}_j}\mu = \beta_j$.

Definition 5.6. The *Brownian linking of the leaves* of foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ passing through the two points x, y in M with respect to the measure μ is the Brownian average $\tilde{f}(x, y)$ of the function

$$f(x, y) := i_{\mathcal{F}(x,y)} LF(x, y)$$

on $M \times M$ along the leaf of the foliation $\mathcal{F}(x, y) = \mathcal{F}_1(x) \times \mathcal{F}_2(y)$ passing through $(x, y) \in M \times M$.

The Brownian average function $\tilde{f}(x, y)$ is constant along the leaves of $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$, i. e., on any given pair of leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$. Note also that the function f measures the linking of the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ just like in the case of linking of compact submanifolds. (If the measures β_i, $i = 1, 2$, are supported on closed non-intersecting surfaces $N_i \subset M$, this is exactly their mutual linking number $\mathrm{lk}(N_1, N_2) = \iint i_{N_1(x)\times N_2(y)} LF(x, y)$, cf. Definition 3.13.)

Theorem 5.7. *Consider a pair of foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of linking dimensions k and l in a compact manifold M. Let ν be a harmonic measure on $M \times M$ for the foliation $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ for any Riemannian metric on M. Then the space average over $M \times M$ for the Brownian ergodic average $\tilde{f}$ of the function $f(x, y) := i_{\mathcal{F}(x,y)} LF(x, y)$ with respect to the measure ν does not depend on the Riemannian metric on M and is equal to the cross-helicity $\mathcal{H}(\mathcal{F}_1, \mathcal{F}_2)$ of the foliations.*

Proof. Fix a Riemannian metric on M and the product metric on $M\times M$. According to the Garnett theorem, there is a harmonic measure ν on $M \times M$.

Consider the Brownian average for the function $f(x,\, y) := i_{\mathcal{F}(x,y)} LF(x,\, y)$ along the leaves of the foliation $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$, where the polyvector $\mathcal{F}$ is defined with the help of the harmonic measure ν: $i_{\mathcal{F}}\nu(x,\, y) = \beta_1(x) \otimes \beta_2(y)$.

The second part of the Garnett theorem immediately gives the required equality:

$$\begin{aligned}\langle \tilde{f},\, \nu\rangle = \iint_{M\times M} \tilde{f}\nu = \iint_{M\times M} f\nu = \iint_{M\times M} (i_{\mathcal{F}} LF)\, \nu \\ = \iint_{M\times M} LF \wedge i_{\mathcal{F}}\nu = \iint_{M\times M} LF \wedge (\beta_1 \otimes \beta_2) = \mathcal{H}(\mathcal{F}_1,\, \mathcal{F}_2).\end{aligned}$$

The latter expression is also equal to $\int_M \beta_1 \wedge d^{-1}\beta_2$ for smooth measures β_1 and β_2. Note that the measure ν is smooth in the latter case, since it can be taken to be the product of the (smooth) transverse holonomy-invariant measure $\beta_1 \otimes \beta_2$ and the Riemannian measure along the leaves of the foliation $\mathcal{F}$. □

Remark 5.8. To give an ergodic interpretation of the higher Hopf invariant (4) one should consider the linking of different leaves of the same foliation defined by the form β. In the latter case one can guarantee that there exists a harmonic form $\bar{\nu}$ on M, such that the harmonic form on $M \times M$ is its square: $\nu = \bar{\nu} \otimes \bar{\nu}$.

For arbitrary transverse holonomy-invariant measures the harmonic measure ν is not smooth in general, cf. [17]. Note that this "Brownian interpretation" does not involve short path systems.

6. Novikov's Integrals

Novikov's invariants [21] can be thought of as a generalization of the Hopf-type invariants to a wide class of maps. In particular, he constructed an analog of the Whitehead operations in the homotopy groups for closed differential forms on manifolds. In [21], Novikov defined a set of invariants on manifolds of an arbitrary dimension, and we consider the four-dimensional case for illustration.

Consider the invariants designed to distinguish various homotopy classes of the maps $S^4 \to \mathbb{R}^3 \setminus \{a,\, b\}$. Their geometric realization in differential forms on S^4 is as follows. Consider a pair of closed 2-forms α and β on S^4 satisfying the following conditions:

$$\alpha \wedge \alpha = \beta \wedge \beta = \alpha \wedge \beta = 0.$$

One can easily check

Proposition 6.1. *The integrals*

$$J(\alpha;\, \alpha,\, \beta) = \int_{S^4} \alpha \wedge d^{-1}\alpha \wedge d^{-1}\beta$$

and

$$J(\beta;\, \alpha,\, \beta) = \int_{S^4} \beta \wedge d^{-1}\alpha \wedge d^{-1}\beta$$

do not depend on the choices of $d^{-1}\alpha$ *and* $d^{-1}\beta$.

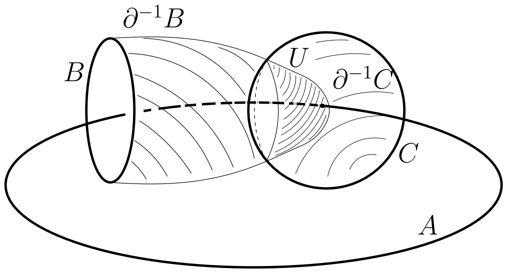

FIGURE 3. The linking of three surfaces A, B, and C in a four-sphere

Remark 6.2. Consider the foliations $\mathcal{A}$ and $\mathcal{B}$ in S^4 defined by the kernel subspaces of the 2-forms α and β. The conditions $\alpha \wedge \alpha = \beta \wedge \beta = 0$ on the pair α, β give the restrictions on the rank of the forms: $\mathrm{rk}(\alpha)$, $\mathrm{rk}(\beta) \leq 2$, so that they define kernel foliations of dimension 2 in S^4. The third condition $\alpha \wedge \beta = 0$ ensures that these kernel foliations have non-transversal intersection. Namely, their leaves at generic points form a 1-dimensional foliation. Moreover, the distribution spanned by the sum of the kernels of α and β is integrable and defines a 3-dimensional foliation [1]. In [16] the integrals above were given an ergodic interpretation in terms of a generalized (non-generic) linking numbers of the corresponding foliations in the spirit of Section 4.3 (see also [17]).

Here we take a somewhat different point of view, inspired by the paper [22]. We discuss below the higher linking numbers related to these integrals, while a Brownian-type ergodic interpretation in terms of linking of the corresponding foliations is achieved by applying the Brownian ergodic theorem [13] discussed in the preceding Section.

More generally, consider the topological meaning of the integral

$$J(\alpha;\, \beta,\, \gamma) = \int_{S^4} \alpha \wedge d^{-1}\beta \wedge d^{-1}\gamma.$$

for a triple of exact 2-forms α, β and γ in S^4, such that $\alpha\wedge\beta = \alpha\wedge\gamma = 0$, providing that it does not depend on the choices of the potential forms $d^{-1}\beta$ and $d^{-1}\gamma$.

Definition 6.3. Let A, B, and C be two-dimensional surfaces in S^4, such that A is disjoint with B and C. Then there is the mutual linking number of such a triple:

$$\mathrm{lk}(A;\, B,\, C) := \#\, A \cap \partial^{-1}B \cap \partial^{-1}C,$$

i. e., the intersection number of A with 3-dimensional films spanning B and C, see Fig. 3.

This definition makes sense for any homological 4-sphere M^4 with $H_1(M) = H_2(M) = 0$. In $\mathbb{R}^4$ there exists also a Gauss-type formula for this linking number.

Definition 6.4 [22]**.** Consider a 6-dimensional manifold $\Sigma^6 := A \times B \times C \subset (\mathbb{R}^4)^3$. Choose points $x \in A$, $y \in B$ and $z \in C$ and normalize the vectors $x - y$ and $x - z$

in $\mathbb{R}^4$. Then we obtain a Gauss type map of 6-dimensional manifolds, from Σ^6 to $S^3 \times S^3 \subset \mathbb{R}^4 \times \mathbb{R}^4$, given by the formula:

$$\rho\colon (x,\, y,\, z) \mapsto \left(\frac{x-y}{|x-y|},\, \frac{x-z}{|x-z|} \right).$$

Its degree can be explicitly written as the integral:

$$\frac{1}{4\pi^4} \iiint_{A\times B\times C} \sum_{i\neq k} \sum_{j\neq l} \frac{x_i - y_i}{|x_i - y_i|^4} \frac{x_j - z_j}{|x_j - z_j|^4}\, dx_k \wedge dx_l \otimes dy_i \wedge dy_k \otimes dz_j \wedge dz_l. \quad (6)$$

The following result was proved jointly with S. Tabachnikov.

Theorem 6.5. (i) *The mutual linking number of a triple is well-defined, i. e., it does not depend on the choices of $\partial^{-1}B$ and $\partial^{-1}C$.*

(ii) *In $\mathbb{R}^4$ this linking number coincides with the degree of the Gauss map*:

$$\mathrm{lk}(A;\, B,\, C) = \deg \rho.$$

Proof. (i) Let A, B and C be the three surfaces in S^4 such that A is disjoint with B and C. If $V = \partial^{-1}B$ and $W = \partial^{-1}C$ are 3-dimensional films spanning B and C, then $V \cap W$ is a surface U with boundary. This boundary does not intersect A, since A is disjoint with B and C, and hence $\#A \cap U$ is well defined. When V is changed to V' with the same boundary B, U is replaced by a homologous U', and therefore, the intersection number $\mathrm{lk}(A;\, B,\, C) = \#A \cap U' = \#A \cap U$ is the same.

(ii) Now suppose that the three initial surfaces are in $\mathbb{R}^4$. Fix two unit vectors, say v and w, and consider the cylinders, say C_v and C_w obtained from B and C by parallel translating them along v and w, respectively. If one moves B and C sufficiently far away in this way, then the mutual linking of A, B, C will be zero. It changed when $C_v \cap C_w$ intersected A. Such an intersection means that there are points $x \in A$, $y \in B$ and $z \in C$, such that the vector $y - x$ is collinear with $-v$ and $z - x$ with $-w$. In other words, this is the degree of the map ρ, computed as the number of preimages of the point $(-v,\, -w) \in S^3 \times S^3$, cf. [22]. □

In order to give the ergodic interpretation of these invariants we need to introduce the corresponding mutual linking form and describe its properties. Define a multi-linking form MLF as a (2, 2, 2)-form on $S^4 \times S^4 \times S^4$ satisfying

$$\iiint_{A\times B\times C \subset S^4\times S^4\times S^4} MLF = \mathrm{lk}(A;\, B,\, C)$$

for any triple of surfaces A, B, C discussed above. Introduce the operator $\widetilde{MLF}\colon$ $\Omega^*(S^4) \times \Omega^*(S^4) \to \Omega^*(S^4)$, acting as follows:

$$\beta(y) \otimes \gamma(z) \mapsto \iint_{S^4_y \times S^4_z} MLF(x,\, y,\, z) \wedge (\beta(y) \otimes \gamma(z)).$$

Remark 6.6. A similar operator $\widetilde{LF}\colon \Omega^*(M) \to \Omega^*(M)$ corresponding to a linking form $LF \in \Omega^*(M \times M)$ on a manifold M sends

$$\beta(y) \mapsto \int_{M_y} LF(x,\, y) \wedge \beta(y).$$

The key property of this operator is that it sends an exact form to its potential: $\widetilde{LF}(\beta) = d^{-1}\beta$. (A linking form is not unique, and so is not a potential.) This property is related to the fact that the differential of the linking form gives the δ-form $\delta(x, y)$ supported on the diagonal $\Delta \subset M \times M$: $LF(x, y) = d_y^{-1}\delta(x, y)$, see details in [2].

Proposition 6.7. (i) *The operator* $\widetilde{MLF}$ *corresponding to a multi-linking form* MLF *sends the product of two exact forms into the product of their potentials*:

$$\widetilde{MLF}\colon \beta(y) \otimes \gamma(z) \mapsto d^{-1}\beta(x) \wedge d^{-1}\gamma(x).$$

In other words, the form $MLF(x, y, z)$ *satisfies the property* $d_y\, d_z MLF(x, y, z) = \delta(x, y)\delta(x, z)$ (*modulo addition of an exact in* x *term*).

(ii) *The multi-linking form* $MLF(x, y, z)$ *can be chosen to be* L^1*-integrable on* $(S^4)^3$.

Proof. (i) By the Stokes formula

$$\iiint_{A\times B\times C} MLF = \iiint_{A\times\partial^{-1}B\times\partial^{-1}C} d_y\, d_z MLF,$$

while by the definition of the multilinking,

$$\iiint_{A\times B\times C} MLF = \mathrm{lk}(A;\, B,\, C) = \iiint_{A\times\partial^{-1}B\times\partial^{-1}C} \delta(x, y)\delta(x, z).$$

Then for the image $\widetilde{MLF}(\beta(y) \otimes \gamma(z))$ we have

$$\begin{aligned}
&\iint_{S^4_y\times S^4_z} MLF(x, y, z) \wedge (\beta(y) \otimes \gamma(z)) \\
&\qquad = \iint_{S^4_y\times S^4_z} d_y\, d_z MLF(x, y, z) \wedge (d_y^{-1}\beta(y) \otimes d_z^{-1}\gamma(z)) \\
&\qquad = \iint_{S^4_y\times S^4_z} \delta(x, y)\delta(x, z) \wedge (d_y^{-1}\beta(y) \otimes d_z^{-1}\gamma(z)) = d^{-1}\beta(x) \wedge d^{-1}\gamma(x).
\end{aligned}$$

(ii) The singularities of the mutual linking form are near the diagonals $x = y$ and $x = z$. Since it is a “local problem,” we can use the Gauss-type formula for $\mathbb{R}^4$. In the latter case integrability is evident from the explicit expression above. □

The integrability of the linking form allows one to substitute the foliations to this form and apply the Brownian ergodic theorem to give the ergodic interpretation of the invariants. Namely, consider the triple of 2-foliations $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ related to a triple of 2-forms α, β and γ of rank 2 on S^4, satisfying the conditions $\alpha\wedge\beta = \alpha\wedge\gamma$.

Theorem 6.8. *Let* ν *be a harmonic measure on* $(S^4)^3$ *for the foliation* $\mathcal{F} = \mathcal{A}\times\mathcal{B}\times\mathcal{C}$ *for any Riemannian metric. Then the space average over* $(S^4)^3$ *for the Brownian ergodic average* $\tilde{g}$ *of the function* $g := i_{\mathcal{F}} MLF$ *with respect to the measure* ν *does not depend on the Riemannian metric on* $(S^4)^3$ *and is equal to the Novikov invariant* $J(\alpha;\, \beta,\, \gamma)$.

The proof mimics the consideration of the preceding section and uses the above two properties of the form MLF:

$$\begin{aligned}\langle \tilde{g}, \nu\rangle &= \iiint_{(S^4)^3} (i_{\mathcal{A}\times\mathcal{B}\times\mathcal{C}} MLF)\nu \\ &= \iiint_{S^4_x\times S^4_y\times S^4_z} MLF(x, y, z)\wedge(\alpha(x)\otimes\beta(y)\otimes\gamma(z)) \\ &= \int_{S^4_x} \alpha(x)\wedge d^{-1}\beta(x)\wedge d^{-1}\gamma(x) = J(\alpha; \beta, \gamma).\end{aligned}$$

Remark 6.9. This theorem was inspired in part by the work [22], where a similar statement was proved under an additional restrictive assumption on the proper behavior of the foliation leaves at infinity (one assumes that the foliation is amenable). In the latter case there is a reformulation of such averaging in terms of closures of the leaves by means of "small caps," similar to the short path approach for trajectories in three dimensions. By lifting this assumption, we have to stick to the Brownian averaging along the leaves of a foliation, without possibility of closing them up, but now such averaging is applicable to a variety of cases, and in particular, to all higher-dimensional versions of the invariants discussed above.

Acknowledgments. The author is grateful for kind hospitality to the ESI in Vienna, the IHES in Bures-sur-Yvette, and Ecole Polytechnique in Paris, where this paper was written. Special thanks go to V. Kaimanovich, M. Kapovich and S. Tabachnikov for illuminating discussions and to D. Novikov for drawing the figures.

References

[1] V. I. Arnold, *The asymptotic Hopf invariant and its applications*, Proc. Summer School in Diff. Equations at Dilizhan (Erevan), 1973, English translation in: Selecta Math. Soviet. **5** (1986), no. 4, 327–345. MR **89m:**58053 (Russian).

[2] V. I. Arnold and B. A. Khesin, *Topological methods in hydrodynamics*, Applied Mathematical Sciences, vol. 125, Springer-Verlag, New York, 1998. MR **99b:**58002

[3] M. A. Berger, *Third-order link invariant*, J. Phys. A **23** (1990), no. 13, 2787–2793. MR **91i:**82023

[4] M. A. Berger and G. B. Field, *The topological properties of magnetic helicity*, J. Fluid Mech. **147** (1984), 133–148. MR **87a:**76118

[5] J. Cantarella, D. DeTurck, and H. Gluck, *The Biot–Savart operator for application to knot theory, fluid dynamics, and plasma physics*, J. Math. Phys. **42** (2001), no. 2, 876–905. MR **2002e:**78002

[6] G. Falkovich, K. Gawędzki, and M. Vergassola, *Particles and fields in fluid turbulence*, Rev. Modern Phys. **73** (2001), no. 4, 913–975. MR **2003f:**76066

[7] V. V. Fock, N. A. Nekrasov, A. A. Rosly, and K. G. Selivanov, *What we think about the higher-dimensional Chern–Simons theories*, Sakharov Memorial Lectures in Physics, Vol. 1, 2 (Moscow, 1991), Nova Sci. Publ., Commack, NY, 1992, pp. 465–471. MR 1 199 342

[8] M. H. Freedman, *Zeldovich's neutron star and the prediction of magnetic froth*, The Arnoldfest (Toronto, ON, 1997), Fields Inst. Commun., vol. 24, Amer. Math. Soc., Providence, RI, 1999, pp. 165–172. MR **2000m:**76116

[9] M. H. Freedman and Z.-X. He, *Divergence-free fields: energy and asymptotic crossing number*, Ann. of Math. (2) **134** (1991), no. 1, 189–229. MR **93a:**58040

[10] J.-M. Gambaudo and É. Ghys, *Enlacements asymptotiques*, Topology **36** (1997), no. 6, 1355–1379. MR **98f:**57050

[11] J.-M. Gambaudo and É. Ghys, *Braids and signatures*, Preprint ENS de Lyon, 2003.

[12] J.-M. Gambaudo and M. Lagrange, *Topological lower bounds on the distance between area preserving diffeomorphisms*, Bol. Soc. Brasil. Mat. (N.S.) **31** (2000), no. 1, 9–27. MR **2002c:**37059

[13] L. Garnett, *Foliations, the ergodic theorem and Brownian motion*, J. Funct. Anal. **51** (1983), no. 3, 285–311. MR **84j:**58099

[14] É. Ghys, personal communication, 2002.

[15] G. Hornig and C. Mayer, *Towards a third-order topological invariant for magnetic fields*, J. Phys. A **35** (2002), no. 17, 3945–3959. MR **2003d:**76164

[16] B. A. Khesin, *Ergodic interpretation of integral hydrodynamic invariants*, J. Geom. Phys. **9** (1992), no. 1, 101–110. MR **93d:**58124

[17] D. Kotschick and T. Vogel, *Linking numbers of measured foliations*, Ergodic Theory Dynam. Systems **23** (2003), no. 2, 541–558. MR 1 972 238

[18] P. Laurence and E. Stredulinsky, *Asymptotic Massey products, induced currents and Borromean torus links*, J. Math. Phys. **41** (2000), no. 5, 3170–3191. MR **2001i:**57010

[19] H. K. Moffatt, *The degree of knottedness of tangled vortex lines*, J. Fluid. Mech. **35** (1969), 117–129.

[20] H. K. Moffatt and A. Tsinober, *Helicity in laminar and turbulent flow*, Annual review of fluid mechanics, Vol. 24, Annual Reviews, Palo Alto, CA, 1992, pp. 281–312. MR **93b:**76018

[21] S. P. Novikov, *The analytic generalized Hopf invariant. Multivalued functionals*, Uspekhi Mat. Nauk **39** (1984), no. 5(239), 97–106 (Russian). MR **86d:**55016. English translation in: Russian Math. Surveys **39** (1984), no. 5, 113–124.

[22] T. Rivière, *High-dimensional helicities and rigidity of linked foliations*, Asian J. Math. **6** (2002), no. 3, 505–533. MR 1 946 345

[23] D. Sullivan, *Cycles for the dynamical study of foliated manifolds and complex manifolds*, Invent. Math. **36** (1976), 225–255. MR 55 #6440

[24] S. L. Tabachnikov, *Two remarks on the asymptotic Hopf invariant*, Funktsional. Anal. i Prilozhen. **24** (1990), no. 1, 84–85 (Russian). MR **91e:**57040. English translation in: Funct. Anal. Appl. **24** (1990), no. 1, 74–75.

[25] T. Vogel, *On the asymptotic linking number*, Proc. Amer. Math. Soc. **131** (2003), no. 7, 2289–2297 (electronic). MR 1 963 779

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