

Symplectic structure on vortex sheets

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To the memory of Vladimir Igorevich Arnold

Abstract

We present a Lie algebraic framework for vortex sheets as singular 2-forms with support of codimension 1, i.e. singular elements of a completion of the dual to the Lie algebra of divergence-free vector fields. This framework allows one to define the Poisson and symplectic structures on the space of vortex sheets, which interpolate between the corresponding structures on filaments and smooth vorticities.

Preface

Vladimir Arnold's 1966 seminal paper [1] in which he introduced numerous geometric ideas into hydrodynamics influenced the field far beyond its original scope. One of his remarkable and, in our opinion, very unexpected Arnold's insights was to regard the fluid vorticity field (or the vorticity 2-form) as an element of the dual to the Lie algebra of the fluid velocities, i.e., the algebra of divergence-free vector fields on the flow domain.

In this paper, after a review of the concept of isovorticed fields, which was crucial, e.g., in Arnold's stability criterion in fluid dynamics, we present an "avatar" of this concept, providing a natural framework for the formalism of vorticity sheets. In particular, we show that the space of vorticity sheets has a natural Poisson structure and occupies an intermediate position between filaments in 3D (or point vortices in 2D) equipped with the Marsden-Weinstein symplectic structure and smooth vorticity fields (vector fields in 3D and scalar fields in 2D) with the Lie-Poisson structure on them.

Before launching into hydrodynamical formalism in this memorial paper, I would like to recall an episode with Vladimir Igorevich, related not to fluid dynamics, but rather to

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his equally surprising insights in real life: his remarks were always witty, to the point, and often mischievous.¹

Back in 1986 Arnold became a corresponding member of the Soviet Academy of Sciences. This was the time of “glasnost” and “acceleration”: novels of many formerly forbidden authors appeared in print for the first time. Jacques Chirac, Prime Minister of France at the time, visited Moscow and gave a speech in front of the Soviet Academy in the Spring of 1986. The speech was typeset beforehand and distributed to the Academy members. Arnold was meeting us, a group of his students, right after Chirac’s speech and brought us that printout. Chirac, who knows Russian, mentioned almost every disgraced poet or writer of the Soviet Russia in his speech: it contained citations from Gumilev, Akhmatova, Mandelstam, Pasternak... And on the top of this printout, above the speech, was the following epigraph in Arnold’s unmistakable handwriting:

*“... Я допущу: успехи наши быстры
Но где ж у нас министр-демагог?
Пусть проберут все списки и регистры,
Я пять рублей бумажных дам в залог;
Быть может, их во Франции немало,
Но на Руси их нет – и не бывало!”*

A.K. Толстой “Сон Попова” (1873)²

Who could take this speech seriously after such a tongue-in-cheek epigraph? As a curious aftermath, Arnold and Chirac shared the Russia State Prize in 2007.

Returning to mathematics, I think that the ideas introduced by Arnold in [1], so natural in retrospect, are in fact most surprising given the state of the art in hydrodynamics of the mid-60s both for their deep insight into the nature of fluids and their geometric elegance and simplicity. In the next section we begin with a brief survey of the use of vorticity in a few hydrodynamical applications and discuss how it helps in understanding the properties of vortex sheets.

¹At times it was hard to tell whether he was being serious or joking. For instance, when inviting the seminar participants for an annual ski trip outside of Moscow, Arnold would say: “This time we are not planning too much, only about 60km. Those who doubt they could make it — need not worry: the trail is so conveniently designed that one can return from interim bus stops on the way, which we’ll be passing by every 20km.”

²English translation by A.B. Givental:

“... Our nation’s rise is, I concur, gigantic,
But demagogues among our statesmen?! Let
all rosters, in the fashion most pedantic,
be searched, I’d put five rubles for a bet:
There could be more than few in France or Prussia,
but are - and have been - none in mother Russia.”

from “Popov’s Dream” by A.K. Tolstoy (1873)

1 The vorticity form of the Euler equation

Consider the Euler equation for an inviscid incompressible fluid filling a Riemannian manifold M (possibly with boundary) equipped with a volume form μ . The fluid motion is described as an evolution of the fluid velocity field v in M which is governed by the classical Euler equation:

$$\partial_t v + v \cdot \nabla v = -\nabla p. \quad (1.1)$$

Here the field v is assumed to be divergence-free ($\operatorname{div} v = 0$) with respect to the volume form μ and tangent to the boundary of M . The pressure function p is defined uniquely modulo an additive constant by these restrictions on the velocity v . The term $v \cdot \nabla v$ stands for the Riemannian covariant derivative $\nabla_v v$ of the field v in the direction of itself.

The vorticity (or Helmholtz) form of the Euler equation is

$$\partial_t \nu + L_v \nu = 0, \quad (1.2)$$

where L_v is the Lie derivative along the field v and which means that the vorticity field $\nu := \operatorname{curl} v$ is transported by the fluid flow. In 3D the vorticity field ν can be thought of as a vector field, while in 2D it is a scalar vorticity function. In the standard 2D-space with coordinates (x_1, x_2) the vorticity function is $\operatorname{curl} v := \partial v_1 / \partial x_2 - \partial v_2 / \partial x_1$, which can be viewed as the vertical coordinate of the vorticity vector field for the 2D plane-parallel flow in 3D. The frozenness of the vorticity allows one to define various invariants of the hydrodynamical Euler equation: e.g., the conservation of helicity in 3D and of enstrophies in 2D.

The Euler equation has the following Hamiltonian formulation. For an n -dimensional Riemannian manifold M with a volume form μ consider the Lie group $G = \operatorname{Diff}_\mu(M)$ of volume-preserving diffeomorphisms of M . The corresponding Lie algebra $\mathfrak{g} = \operatorname{Vect}_\mu(M)$ consists of divergence-free vector field in M : $\operatorname{Vect}_\mu(M) = \{v \in \operatorname{Vect}(M) \mid L_v \mu = 0\}$. The natural dual space for this Lie algebra is the space of cosets of 1-forms on M modulo exact 1-forms, $\mathfrak{g}^* = \Omega^1(M) / \Omega^0(M)$. The pairing between cosets of 1-forms $[u]$ and vector fields $w \in \operatorname{Vect}_\mu(M)$ is given by $\langle [u], w \rangle := \int_M i_w u \cdot \mu$. The Euler equation (1.1) on the dual space assumes the form

$$\partial_t [u] + L_v [u] = 0,$$

where $[u] \in \Omega^1(M) / \Omega^0(M)$ stands for the coset of the 1-form $u = v^\flat$ related to the vector field v by means of the Riemannian metric on M : $u = v^\flat$. (For a manifold M equipped with a Riemannian metric (\cdot, \cdot) one defines the 1-form v^\flat as the pointwise inner product with vectors of the velocity field v : $v^\flat(\eta) := (v, \eta)$ for all $\eta \in T_x M$, see details in [1, 4].)

Instead of dealing with cosets of 1-forms, it is often more convenient to pass to their differentials. The *vorticity 2-form* $\xi := dv^\flat$ is the differential of the 1-form $u = v^\flat$. Note that in 3D the vorticity vector field $\nu = \operatorname{curl} v$ is defined by the 2-form ξ via $i_\nu \mu := \xi$ for the volume form μ . In 2D $\nu = \operatorname{curl} v$ is the function $\nu := \xi / \mu$. The definition of vorticity ξ as an exact 2-form in M makes sense for any dimension of the manifold M . This point of view can be traced back to the original papers by Arnold, see e.g. [3, 2]

Such a definition immediately implies that: *i*) the vorticity 2-form is well-defined for cosets $[u]$: 1-forms at the same coset have equal vorticities, and *ii*) the Euler equation in the

form (1.2) or $\partial_t(du) + L_v(du) = 0$ means that the vorticity 2-form $\xi = du$ is transported by (or frozen into) the fluid flow. This frozenness of the vorticity 2-form allows one to define invariants similar to enstrophies for all even-dimensional flows and helicity-type integrals for all odd-dimensional ideal fluid flows, which turn out to be first integrals of the corresponding higher-dimensional Euler equation, see e.g. [4].

Remark 1.1. Two main applications of such a point of view on vorticity is the existence of the Poisson structure and Arnold's stability criterion. Assume that $H^1(M) = 0$ to simplify the reasoning below. Then the space of vorticities $\{\xi\}$, i.e. the space of exact 2-forms $d\Omega^1(M)$, coincides with the dual space to the Lie algebra $\text{Vect}_\mu(M)$ of divergence-free vector fields. Indeed, $\text{Vect}_\mu(M)^* \simeq \Omega^1/d\Omega^0 \simeq d\Omega^1$ where the latter identification holds since $H^1(M) = 0$.

As the dual space, the vorticity space $\{\xi\} = \text{Vect}_\mu(M)^*$ has the natural Lie-Poisson structure. Moreover, sets of isovorticed fields turns out to be coadjoint orbits of the corresponding group $\text{Diff}_\mu(M)$, since the group action on the vorticity is geometric, by the diffeomorphism action on a 2-form. These sets of isovorticed fields are symplectic submanifolds in the dual $\mathfrak{g}^* = \text{Vect}_\mu(M)^*$ and the Euler equation defines a Hamiltonian evolution on them.

Remark 1.2. This was the basis for Arnold's stability criterion. Namely, steady fluid flows are critical points of the restriction of the Hamiltonian (which is the energy function on the dual space) to the coadjoint orbits, i.e. to isovorticed fields. If the restriction of the Hamiltonian functional has a sign-definite (positive or negative) second variation at the critical point, the corresponding steady flow is Lyapunov stable. This is the famous Arnold's stability test. In particular, he proved (see e.g. [3]) that shear flows in an annulus with no inflection points in the velocity profile are Lyapunov stable, thus generalizing the Raleigh stability condition.

2 Singular vorticities in codimension 2: point vortices and filaments

Let M be an n -dimensional manifold with a volume form μ and filled with an incompressible fluid. As we discussed above, the vorticity of any fluid motion is geometrically an exact 2-form. For a smooth vector field u it is defined as $\xi := dv^b$, where v^b is the 1-form obtained from the vector field v by the metric lifting of indices.

Regular vorticities have support of $\text{codim}=0$, while singular ones have support of $\text{codim} \geq 1$. Singular vorticities form a subspace in (a completion of) the dual space $\mathfrak{g}^* = d\Omega^1(M)$. Note that since vorticity is a (possibly singular) 2-form (more precisely, Green current of degree 2), its support has to be of $\text{codim} \leq 2$. (E.g., if support is of $\text{codim}=3$, it corresponds to a singular 3-form.)

Most interesting cases of support are of codimension 1 (vortex sheets) and codimension 2 (filaments in 3D and point vortices in 2D). In this section we consider the codimension 2 case.

A. Point vortices.

Let the symplectic vorticity ν be supported on N point vortices: $\nu = \sum_{j=1}^N \Gamma_j \delta(z - z_j)$, where $z_j = (x_j, y_j)$ are coordinates of the j th point vortex in the Euclidean space $\mathbb{R}^2 = \mathbb{C}^1$ with the standard area form $\omega = dx \wedge dy$. Kirchhoff's theorem states that the evolution of vortices according to the Euler equation is described by the system

$$\Gamma_j \dot{x}_j = \frac{\partial \mathcal{H}}{\partial y_j}, \quad \Gamma_j \dot{y}_j = -\frac{\partial \mathcal{H}}{\partial x_j}, \quad 1 \leq j \leq N. \quad (2.3)$$

This is a Hamiltonian system on \mathbb{R}^{2N} with the Hamiltonian function

$$\mathcal{H} = -\frac{1}{4\pi} \sum_{j < k}^N \Gamma_j \Gamma_k \ln |\tilde{z}_j - \tilde{z}_k|^2$$

and the Poisson structure is given by the bracket

$$\{f, g\} = \sum_{j=1}^N \frac{1}{\Gamma_j} \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial y_j} - \frac{\partial f}{\partial y_j} \frac{\partial g}{\partial x_j} \right). \quad (2.4)$$

One can derive the above Hamiltonian dynamics from the 2D Euler equation in the vorticity form $\partial_t \nu = \{\psi, \nu\}$, where ν is the vorticity function in \mathbb{R}^2 and the stream function (or Hamiltonian) ψ of the flow satisfies $\Delta \psi = \nu$, see e.g. [7]. While the system (2.3) goes back to Kirchhoff, its properties for various numbers of point vortices and versions for different manifolds have been of constant interest, see e.g. [8, 7]. The cases of $N = 2$ and $N = 3$ vortices are integrable, while those of $N \geq 4$ are not.

Here we would like to point out the origin of the Poisson bracket (2.4).

Proposition 2.1. *The Poisson bracket (2.4) is defined by the Kirillov-Kostant symplectic structure on the coadjoint orbit of the (singular) vorticity $\nu = \sum_{j=1}^N \Gamma_j \delta(z - z_j)$ in (the completion of) the dual of the Lie algebra $\mathfrak{g} = \text{Vect}_\mu(\mathbb{R}^2)$ of divergence free vector fields in \mathbb{R}^2 .*

Proof. Indeed, a vector tangent to the coadjoint orbit of such a singular vorticity ν can be regarded as a collection of vectors in \mathbb{R}^2 attached at z_j . Then for a pair of tangent vectors the corresponding Kirillov-Kostant symplectic structure becomes the weighted sum of the corresponding contributions at each point vortex z_j . The pairing with the momentum ν , which appears in the Kirillov-Kostant bracket, gives the strength Γ_j as the corresponding weight. The Poisson bracket (2.4), being the inverse of the symplectic structure, has the reciprocals of the weights. QED.

B. Filaments in 3D.

While moving from 2D to 3D we pass from point vortices to filaments. Filaments are curves in \mathbb{R}^3 being supports of singular vorticity fields. Vorticity filaments are governed by the Euler equation

$$\partial_t \xi + L_v \xi = 0, \quad (2.5)$$

where $v = \text{curl}^{-1} \xi$ and the vorticity field (or a 2-form) ξ has support on a curve $\gamma \subset \mathbb{R}^3$. (Note that the exactness of the field ξ implies that γ is a boundary of a 2-dimensional domain, i.e., in particular, its components are either closed or go to infinity.) This Euler dynamics of the vorticity field or 2-form is nonlocal, since it requires finding the field curl^{-1} .

The localized induction approximation (LIA) of the vorticity motion is a procedure which allows one to keep only the local terms in the equation, as we discuss below. In \mathbb{R}^3 the corresponding evolution describes the filament equation

$$\partial_t \gamma = \gamma' \times \gamma'', \quad (2.6)$$

where $\gamma(\cdot, t) \subset \mathbb{R}^3$ is a time-dependent arc-parametrized space curve. For an arbitrary parametrization the filament equation becomes $\partial_t \gamma = \kappa \cdot \mathbf{b}$, where κ and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ stand, respectively, for the curvature and binormal unit vector of the curve γ at the corresponding point. This equation is often called the binormal equation.

Here we briefly recall the LIA derivation of the equation (2.6), see e.g. [5]. Assume that the velocity distribution v in an unbounded simply connected domain M in \mathbb{R}^3 has vorticity $\xi = \text{curl} v$ concentrated on a smooth arc-length parametrized curve of length L . Then

$$\xi(x, t) = C \int_0^L \delta(x - \gamma(\theta, t)) \frac{\partial \gamma}{\partial \theta} d\theta.$$

Here δ is the δ -function in \mathbb{R}^3 and the constant C is the flux of ξ across (or, that is the same, the circulation of v over) a small contour around the core of the filament.

The Biot-Savart law allows one to represent the velocity field in terms of its vorticity:

$$v(x, t) = -\frac{1}{4\pi} \int_M \frac{(x - \tilde{x}) \times \xi(\tilde{x})}{\|x - \tilde{x}\|^3} d^3 \tilde{x} = -\frac{C}{4\pi} \int_\gamma \frac{x - \gamma(\tilde{\theta}, t)}{\|x - \gamma(\tilde{\theta}, t)\|^3} \times \frac{\partial \gamma}{\partial \tilde{\theta}} d\tilde{\theta}.$$

By utilizing the fact that the time evolution of the curve γ is given by the velocity field v itself: $\frac{\partial \gamma}{\partial t}(\theta, t) = v(\gamma(\theta, t), t)$ we come to the following integral:

$$\frac{\partial \gamma}{\partial t}(\theta, t) = -\frac{C}{4\pi} \int_\gamma \frac{\gamma(\theta, t) - \gamma(\tilde{\theta}, t)}{\|\gamma(\theta, t) - \gamma(\tilde{\theta}, t)\|^3} \times \frac{\partial \gamma}{\partial \tilde{\theta}} d\tilde{\theta}.$$

This integral is divergent with the main singularity coming from the points on the curve γ close to each other on the curve (with small $\|\tilde{\theta} - \theta\|$). The Taylor expansion of $\gamma(\theta)$ in $(\tilde{\theta} - \theta)$ yields

$$\frac{\partial \gamma}{\partial t}(\theta, t) = \frac{C}{8\pi} \left[\frac{\partial \gamma}{\partial \theta} \times \frac{\partial^2 \gamma}{\partial \theta^2} \int_0^L \frac{d\tilde{\theta}}{|\theta - \tilde{\theta}|} + \mathcal{O}(1) \right]$$

as $\tilde{\theta} \rightarrow \theta$. By keeping only the singularity term and neglecting others we come to the localized induction approximation: only parts of the curve sufficiently close to a given point θ determine the velocity field at that point. By introducing the cut-off beyond $|\theta - \tilde{\theta}| > \epsilon$ and rescaling time one obtains the vortex filament equation (2.6).

The passage to the binormal form is straightforward: for an arc-parametrization the tangent vectors $\mathbf{t} = \partial\gamma/\partial\theta = \gamma'$ have unit length and the acceleration vectors are $\gamma'' = \partial\mathbf{t}/\partial\theta = \kappa \cdot \mathbf{n}$, i.e. $\partial_t\gamma = \gamma' \times \gamma''$ becomes $\partial_t\gamma = \kappa \cdot \mathbf{b}$, where the latter equation is valid for an arbitrary parametrization.

Remark 2.2. This binormal equation is known to be Hamiltonian relative to the Marsden-Weinstein symplectic structure on non-parametrized space curves in \mathbb{R}^3 .

Recall that the Marsden-Weinstein symplectic structure is defined on curves γ by

$$\Omega_\gamma(v, w) := \int \det(\gamma', v, w) d\theta \quad (2.7)$$

where v and w are two vector fields attached to the curve γ and regarded as variations of γ . Equivalently, this symplectic structure can be defined by means of the operator \mathbf{J} of almost complex structure on curves, which makes a skew-gradient from a gradient field: any variation, i.e. vector field attached at the curve γ is rotated by the operator \mathbf{J} in the planes orthogonal to γ by $\pi/2$ in the positive direction, see details in [10, 4]. One can show that this is the Kirillov-Kostant symplectic structure on the coadjoint orbit of the curve $\gamma = \text{supp } \xi$, understood as a point in the dual of the Lie algebra: $\xi \in d\Omega^1(\mathbb{R}^3) = \text{Vect}_\mu(\mathbb{R}^3)^*$.

The pairing of γ and a divergence-free vector field v can be defined directly as $\langle \gamma, v \rangle := \text{Flux } v|_\sigma$, where σ is an oriented surface whose boundary is $\gamma = \partial\sigma$.

As discussed above the Euler equation (2.5) is Hamiltonian with the Hamiltonian function given by the kinetic energy. The energy $E(v) = \langle v, v \rangle / 2$ is local in terms of velocity fields, but it is nonlocal in terms of vorticities: $E(v) = \frac{1}{2} \langle \text{curl}^{-1}\xi, \text{curl}^{-1}\xi \rangle / 2$. It turns out that after taking the localized induction approximation, when we kept only the local terms, the filament equation remains Hamiltonian with respect to the same Marsden-Weinstein symplectic structure, but with a different Hamiltonian.

The corresponding Hamiltonian functional turns out to be the length functional of the curve: $H(\gamma) = \text{length}(\gamma) = \int_\gamma \|\gamma'(\theta)\| d\theta$, see e.g. [4]. Indeed, the variational derivative, i.e. the “gradient,” of this length functional H is $\delta H / \delta \gamma = -\gamma'' = -\mathbf{t}' = -\kappa \cdot \mathbf{n}$, where \mathbf{t} and \mathbf{n} are, respectively, the unit tangent and normal fields to the curve γ . The dynamics is given by the corresponding skew-gradient, which is obtained from $\delta H / \delta \gamma$ by applying \mathbf{J} for the above symplectic structure. This operator rotating in the plane orthogonal to \mathbf{t} sends $-\kappa \cdot \mathbf{n}$ to $\kappa \cdot \mathbf{b}$.

The Marsden-Weinstein symplectic structure also exists on the spaces of immersed submanifolds of codimension 2 in M of any dimension. In a similar way one defines an analog of the Hamiltonian functional (the area of the corresponding submanifolds) and the higher-dimensional analog of the binormal equation, see [6].

3 Singular vorticities in codimension 1: vortex sheets

A. *Vortex sheets as singular exact 2-forms.*

Now we confine ourselves to singular vorticities supported in codimension 1.

Definition 3.1. *Vortex sheets* are singular exact 2-forms, i.e. Green 2-currents of type $\xi = \alpha \wedge \delta_\Gamma$, where $\Gamma^{n-1} \subset M^n$ is a closed oriented hypersurface in M , δ_Γ is the corresponding Dirac 1-current, and α is a closed 1-form on Γ .

For a singular 2-form $\xi = \alpha \wedge \delta_\Gamma$ to be exact either the closed 1-form α must be exact on Γ , i.e. $\alpha = df$ for a function f on Γ , or the hypersurface Γ must be a boundary of some domain $\partial^{-1}\Gamma \subset M$ and the closed 1-form α has to admit an extension to a closed 1-form A on $\partial^{-1}\Gamma$. (Note that under the assumption $H^1(M) = 0$ a closed hypersurface Γ is always a boundary.) Indeed, the wedge product of exact and closed forms is exact.

For an exact form $\alpha = df$ the vortex sheet is fibered by levels of the function f . If α is a closed 1-form, it is a function differential only locally, and the integral submanifolds of $\ker \alpha$ foliate Γ . Thus the vortex sheet is fibered into filaments (of codimension=1 in Γ) in the former case and foliated in the latter.

Example 3.2. If α is supported on a single hypersurface γ in Γ (i.e. on a curve $\gamma \subset \Gamma$ for $n = 3$), then the vortex sheet $\xi = \alpha \wedge \delta_\Gamma = \delta_\gamma$ reduces to the vorticity of the filament $\gamma \subset \Gamma$.

Remark 3.3. The corresponding primitive 1-forms u satisfying $\xi = du$ for the singular vorticity 2-form $\xi = df \wedge \delta_\Gamma$ are as follows. For an exact $\alpha = df$ take $u = f\delta_\Gamma$. On the other hand, for a closed 1-form α extendable to a closed 1-form A on a domain $\partial^{-1}\Gamma$ take a primitive $u = d^{-1}\xi$ to be $u = -\chi_{\partial^{-1}\Gamma} \cdot A$, where $\partial^{-1}\Gamma$ is a domain bounded by the hypersurface Γ and $\chi_{\partial^{-1}\Gamma}$ is its characteristic function.

Indeed,

$$du = -d(\chi_{\partial^{-1}\Gamma} \cdot A) = -d\chi_{\partial^{-1}\Gamma} \wedge A = -\delta_\Gamma \wedge A = \alpha \wedge \delta_\Gamma = \xi.$$

Note that the 1-form A and the domain $\partial^{-1}\Gamma$ are not defined uniquely, and this ambiguity corresponds to the ambiguity in the definition of a primitive $u = d^{-1}\xi$.

Such singular vorticity currents ξ can be regarded as elements of the dual space $\text{Vect}_\mu(M)^*$.

Definition–Proposition 3.4. *The pairing of vortex sheets (i.e. singular vorticity currents) $\xi = \alpha \wedge \delta_\Gamma$ with vector fields $v \in \text{Vect}_\mu(M)$ is defined by*

$$\langle \langle \xi, v \rangle \rangle := \langle \text{curl}^{-1}(\alpha \wedge \delta_\Gamma), v \rangle = \int_M i_v \text{curl}^{-1}(\alpha \wedge \delta_\Gamma) \cdot \mu,$$

where $u = \text{curl}^{-1}(\alpha \wedge \delta_\Gamma)$ is a primitive 1-form for the vorticity $\xi = \alpha \wedge \delta_\Gamma$. The above pairing for an exact $\alpha = df$ reduces to

$$\langle \text{curl}^{-1}(df \wedge \delta_\Gamma), v \rangle = \text{Flux}(fv)|_\Gamma.$$

For a 1-form α extendable to a 1-form A on $\partial^{-1}\Gamma$ the pairing is

$$\langle \text{curl}^{-1}(\alpha \wedge \delta_\Gamma), v \rangle = \int_{\partial^{-1}\Gamma} i_v A \cdot \mu.$$

In either case the pairing is well-defined, i.e. it does not depend on the choice of curl^{-1} .

Proof. In the exact case $\alpha = df$ one needs to check that the pairing does not depend on the choice of a function f . Such functions can differ by an additive constant: $\tilde{f} = f + C$. Then

$$\text{Flux}(\tilde{f}v)|_\Gamma = \text{Flux}(fv)|_\Gamma + C \cdot \text{Flux}(v)|_\Gamma = \text{Flux}(fv)|_\Gamma,$$

since $\text{Flux}(v)|_\Gamma = 0$ (as the flux of a divergence-free field v across a closed surface Γ vanishes).

For an extendable α two different extensions A and $\tilde{A} = A + B$ differ by a closed 1-form B on $\partial^{-1}\Gamma$ which vanishes on Γ itself. Since one can extend this 1-form B by zero outside of $\partial^{-1}\Gamma$, we see that B is a closed and hence exact 1-form on the whole of M , $B = dF$, since $H^1(M) = 0$. Then

$$\int_{\partial^{-1}\Gamma} i_v \tilde{A} \cdot \mu = \int_{\partial^{-1}\Gamma} i_v A \cdot \mu + \int_{\partial^{-1}\Gamma} i_v B \cdot \mu = \int_{\partial^{-1}\Gamma} i_v A \cdot \mu + \int_M i_v dF \cdot \mu = \int_{\partial^{-1}\Gamma} i_v A \cdot \mu,$$

since $\int_{\partial^{-1}\Gamma} (i_v dF) \mu = \int_{\partial^{-1}\Gamma} (L_v F) \mu = \int_{\partial^{-1}\Gamma} F \cdot L_v \mu = 0$ due to the divergence-free property of v . QED

Remark 3.5. Now we discuss how to define the vortex sheet $\xi = \alpha \wedge \delta_\Gamma$, i.e. the 1-form α on Γ , if Γ is the oriented boundary between two different parts M_j with velocity fields v_1, v_2 that are divergence-free and vorticity-free (i.e. potential flows).

Given a Riemannian metric on M we prepare the 1-form v_j^\flat on M_j , corresponding to the velocity v_j respectively. Note that the forms v_j^\flat must be locally exact, $v_j^\flat = dF_j$ since $\text{curl } v_j = 0$ on M_j . Then locally $\alpha := (dF_1 - dF_2)|_\Gamma = df_1 - df_2$. One can also define this 1-form $\alpha = d(f_1 - f_2)$ by means of the vector field v_Γ inside this vortex sheet Γ by using the metric restricted to Γ : locally $v_\Gamma := (d(f_1 - f_2))^\sharp = \text{Proj}|_\Gamma(v_1 - v_2)$. The proper sign of v_Γ or the form α depends on the orientation of Γ : the latter defines the orientation of the corresponding exterior normal and hence signs of the fields v_1 and v_2 in this difference.

B. Definition and properties of the symplectic structure on vortex sheets.

There is a natural Poisson structure on vortex sheets, coming from $\text{Vect}_\mu(M)^*$ and generalizing the Marsden-Weinstein symplectic structure for filaments in \mathbb{R}^3 or, more generally, for submanifolds of codimension 2 in M . The corresponding symplectic leaves are defined by isovorticed fields, i.e. fields with diffeomorphic singular vorticities $\alpha \wedge \delta_\Gamma$. We describe the corresponding symplectic structure on spaces of diffeomorphic vortex sheets.

Definition 3.6. Given two vector fields U, V attached at Γ define the symplectic structure on variations of vorticity sheets $\alpha \wedge \delta_\Gamma$, i.e. pairs (Γ, α) , by

$$\Omega_\xi(V, W) := \int_\Gamma \alpha \wedge i_V i_W \mu.$$

Theorem 3.7. *The form Ω_ξ coincides with the Kirillov-Kostant symplectic structure ω_{KK} on the coadjoint orbit containing the vorticity ξ in $\text{Vect}_\mu^*(M)$.*

Proof. Let V and W be two divergence-free vector fields in M , which we regard as a pair of variations of the point ξ in $\text{Vect}_\mu(M)^*$. The Kirillov-Kostant symplectic structure on the coadjoint orbit of the vortex sheet $\xi = \delta_\Gamma \wedge \alpha$ associates to a pair of variations tangent to the coadjoint orbit of ξ the following quantity:

$$\omega_{KK}(\xi)(V, W) := \langle d^{-1}\xi, [V, W] \rangle = \langle u, [V, W] \rangle = \int_M u \wedge i_{[V, W]}\mu.$$

Here $[V, W]$ is the commutator of the vector fields V and W , and the 1-form $u = d^{-1}\xi$ is a primitive of the singular 2-form ξ . Note that the commutator of divergence-free fields V and W with respect to the volume form μ satisfies the identity $i_{[V, W]}\mu = d(i_V i_W \mu)$.

Then

$$\begin{aligned} \omega_{KK}(\xi)(V, W) &= \int_M u \wedge d(i_V i_W \mu) = \int_M du \wedge i_V i_W \mu = \int_M \xi \wedge i_V i_W \mu \\ &= \int_M \delta_\Gamma \wedge \alpha \wedge i_V i_W \mu = \int_\Gamma \alpha \wedge i_V i_W \mu = \Omega_\xi(V, W). \end{aligned}$$

QED

Remark 3.8. If α is supported on a curve $\gamma \subset \Gamma$, i.e. $\xi = \delta_\gamma$, then $\Omega_\xi(U, V) := \int_\gamma i_U i_V \mu$. For a curve $\gamma \subset \mathbb{R}^3$ this is exactly the Marsden-Weinstein symplectic structure (2.7) on filaments, i.e. non-parametrized curves in \mathbb{R}^3 .

The evolution of vortex sheets $\xi = df \wedge \delta_\Gamma$ is defined by the classical Euler equation in the vorticity form $\partial_t \xi + L_v \xi = 0$, where $\xi = \text{curl } v$. This equation is Hamiltonian with respect to the above symplectic structure Ω_ξ . The standard energy Hamiltonian $E(v) = \langle v, v \rangle / 2$, as above, defines a non-local evolution of the vorticity sheet.

Let (f, θ) be coordinates on a vortex sheet $\alpha \wedge \delta_\Gamma$ in \mathbb{R}^3 where the exact 1-form $\alpha = df$ and the surface Γ is fibrated into the filaments Γ_f being the levels of the function f . The rough LIA procedure similar to the one described in Section 2 under the assumption $|f - \tilde{f}|^2 \leq |\theta - \tilde{\theta}|$ leads to the binormal type equation: $\partial_t \Gamma = \Gamma_\theta \times \Gamma_{\theta\theta}$, which is Hamiltonian with the Hamiltonian function $H(\Gamma) = \int \text{length}(\Gamma_f) df$. The latter is a continuous family of binormal equations. One may hope that other assumptions on the cut-off procedure lead to more interesting approximations.

Problem 3.9. What are analogs of more precise localized induction approximations (LIAs) and the length Hamiltonian for vortex sheets?

Note that just like for point vortices and filaments, the initial position $\xi(0)$ of vortex sheets defines their consequent evolution, i.e. they “do not have inertia” (unlike the many-body problem, where one needs to prescribe both initial positions and velocities). The motion of vortex sheets is known to be very unstable [9]. It would be interesting to obtain this instability within the Hamiltonian framework for vortex sheets described above.

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