DISCRETE INVARIANTS OF GENERICALLY INCONSISTENT SYSTEMS OF LAURENT POLYNOMIALS

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Abstract. Consider a system of equations

$$f_1 = \cdots = f_k = 0$$

in $(\mathbb{C}^*)^n$, where $f_1, \ldots, f_k$ are Laurent polynomials with the supports in $\mathcal{A}_1, \ldots, \mathcal{A}_k \subset \mathbb{Z}^n$. Assume that the generic system with fixed supports $\mathcal{A}_1, \ldots, \mathcal{A}_k$ is inconsistent. One can ask how to compute discrete invariants of $Y \subset (\mathbb{C}^*)^n$ defined by a system of equations which is generic **in the set of consistent systems** with supports in $\mathcal{A}_1, \ldots, \mathcal{A}_k$. In this paper we show how to solve this problem by reducing it to the Newton polyhedra theory. Unlike the classical situation, not only the Newton polyhedra of $f_1, \ldots, f_k$, but also the supports $\mathcal{A}_1, \ldots, \mathcal{A}_k$ themselves appear in the answers.

1. Introduction.

With a Laurent polynomial f in n variables one can associate its support $sup(f) \subset \mathbb{Z}^n$ which is the set of exponents of monomials having non-zero coefficient in f and its Newton polyhedra $\Delta(f) \subset \mathbb{R}^n$ which is the convex hull of the support of f in $\mathbb{R}^n$. Consider an algebraic variety $Y \subset (\mathbb{C}^*)^n$ defined by a system of equations

$$f_1 = \cdots = f_k = 0, \tag{1}$$

where $f_1, \ldots, f_k$ are Laurent polynomials with the supports in finite sets $\mathcal{A}_1, \ldots, \mathcal{A}_k \subset \mathbb{Z}^n$. Assume that $f_1, \ldots, f_k$ are generic equations with supports in $\mathcal{A}_1, \ldots, \mathcal{A}_k$. The Newton polyhedra theory computes the discrete invariants of Y in terms of the Newton polyhedra $\Delta_1, \ldots, \Delta_k$. One of the first examples of such result is the Bernstein-Kouchnirenko theorem.

Theorem 1 (Bernstein-Kouchnirenko). *Let $f_1, \ldots, f_n$ be generic Laurent polynomials with supports in $\mathcal{A}_1, \ldots, \mathcal{A}_n$. Then all solutions of the system $f_1 = \ldots = f_n = 0$ in $(\mathbb{C}^*)^n$ are non-degenerate and the number of them is equal to*

$$n! Vol(\Delta_1, \ldots, \Delta_n),$$

where Δ_i is the convex hull of $\mathcal{A}_i$ and Vol is the mixed volume.

For some of other examples see [DKh],[Kh],[Kh2].

The Newton polyhedra theory computes invariants of Y assuming that the system (1) is generic enough. That is, there exists a proper algebraic subset Σ in the space Ω of coefficients of $f_1, \ldots, f_k$ such that the corresponding discrete invariant

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is constant in $\Omega \setminus \Sigma$ and could be computed in terms of polyhedra $\Delta_1, \dots, \Delta_k$. If $(f_1, \dots, f_k) \in \Sigma$, the invariants of Y depend not only on $\Delta_1, \dots, \Delta_k$ and are much harder to compute.

In the case that $\mathcal{A}_1, \dots, \mathcal{A}_k$ are such that the general system is inconsistent in $(\mathbb{C}^*)^n$ one can modify the question in the following way. **What are discrete invariants of a zero set of generic consistent system with given supports?** The main result of this paper is Theorem 6 which reduces this question to the Newton Polyhedra theory. In this situation, the discrete invariants are computed in terms of supports themselves, not the Newton polyhedra. Some examples of applications of Theorem 6 are given in Section 5.

2. Preliminary facts on the Set of consistency.

The material of this section is well-known (see for example [GKZ], [St], [D'AS]).

2.1. **Definition of the Incidence variety and the Set of consistency.** Let $A = (\mathcal{A}_1, \dots, \mathcal{A}_k)$ be a collection of k finite subsets of the lattice $\mathbb{Z}^n$. The space Ω_A of Laurent polynomials $f_1, \dots f_k$ with supports in $\mathcal{A}_1, \dots, \mathcal{A}_k$ is isomorphic to $(\mathbb{C})^{|\mathcal{A}_1|+\dots+|\mathcal{A}_k|}$, where $|\mathcal{A}_i|$ is the number of points in $\mathcal{A}_i$.

Definition 1. The *incidence variety* $\tilde{X}_A \subset (\mathbb{C}^*)^n \times \Omega_A$ is defined as:

$$\tilde{X}_A = \{(p, (f_1, \dots, f_k)) \in (\mathbb{C}^*)^n \times \Omega_A | f_1(p) = \dots = f_k(p) = 0\}.$$

Let $\pi_1 : (\mathbb{C}^*)^n \times \Omega_A \to (\mathbb{C}^*)^n$, $\pi_2 : (\mathbb{C}^*)^n \times \Omega_A \to \Omega_A$ be natural projections to the first and the second factors of the product.

Definition 2. The *set of consistency* $X_A \subset \Omega_A$ is the image of $\tilde{X}_A$ under the projection π_2.

Theorem 2. *The incidence variety $\tilde{X}_A \subset (\mathbb{C}^*)^n \times \Omega_A$ is a smooth algebraic variety.*

Proof. Indeed, the projection π_1 restricted to the $\tilde{X}_A$:

$$\pi_1 : \tilde{X}_A \to (\mathbb{C}^*)^n$$

form a vector bundle of rank $(\mathbb{C})^{|\mathcal{A}_1|+\dots+|\mathcal{A}_k|-k}$. That is because for the fixed point $p \in (\mathbb{C}^*)^n$ the preimage $\pi_1^{-1}(p) \subset \tilde{X}_A$ is given by k obviously independent linear equations on the coefficients of polynomials $f_1, \dots, f_k$. $\square$

We will say that the semi-algebraic subset X of $\mathbb{C}^N$ is irreducible if for any two polynomials f, g such that $fg|_X = 0$ either $f|_X = 0$ or $g|_X = 0$.

Corollary 1. *The set of consistency X_A is an irreducible semi-algebraic subset of Ω_A.*

Proof. Since $X_A = \pi_2(\tilde{X}_A)$ is the image of an irreducible algebraic variety $\tilde{X}_A$ under the algebraic map π_2, it is semi-algebraic and irreducible. $\square$

2.2. **Codimension of the set of consistency.** We will start this section with a definition. For the collection $B = (\mathcal{B}_1, \ldots, \mathcal{B}_l)$ of finite subsets of $\mathbb{Z}^n$ let $\mathcal{B} = \mathcal{B}_1 + \ldots + \mathcal{B}_l$ be the Minkowskii sum of all subsets in the collection and let $L(B)$ be the linear subspace parallel to the minimal affine subspace containing $\mathcal{B}$.

Definition 3. The defect of a collection $B = (\mathcal{B}_1, \ldots, \mathcal{B}_l)$ of finite subsets of $\mathbb{Z}^n$ is given by

$$def(\mathcal{B}_1, \ldots, \mathcal{B}_l) = dim(L(B)) - l.$$

For a subset $J \subset \{1, \ldots, l\}$ let us define the collection $B_J = (\mathcal{B}_i)_{i \in J}$. For the simplicity we denote the defect $def(B_J)$ by $def(J)$, and the linear space $L(B_J)$ by $L(J)$.

The following theorem provides a criterion for a system of Laurent polynomials with supports in $\mathcal{A}_1, \ldots, \mathcal{A}_k$ to be generically consistent.

Theorem 3 (Bernstein). *A system of generic equations $f_1 = \ldots = f_k = 0$ of Laurent polynomials with supports in $\mathcal{A}_1, \ldots, \mathcal{A}_k$ respectively has a common root if and only if for any $J \subset \{1, \ldots, k\}$ the defect $def(J)$ is nonnegative.*

According to the Bernstein theorem if there exist subcollection of A with negative defect the codimension of the set of consistency is positive. We will call such collections A *generically inconsistent.* The following theorem of Sturmfelds determines precise codimension of X_A.

Theorem 4 (Sturmfelds). *Let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be such that the generic system with supports in $\mathcal{A}_1, \ldots, \mathcal{A}_k$ is inconsistent. Then the codimension of the set of consistency X_A in Ω_A is equal to the maximum of the numbers $-def(J)$, where J runs over all subsets of $\{1, \ldots, k\}$.*

For $\mathcal{A}_1, \ldots, \mathcal{A}_k$ – a collection of finite subsets of $\mathbb{Z}^n$ we will denote by $d(\mathcal{A})$ the smallest defect $def(J)$ of the subcollection $J \subset \{1, \ldots, k\}$. Suppose a collection A is generically inconsistent i.e. $d(A) < 0$. We will call a subcollection J *essential* if $def(J) = d$ and $def(I) > d$ for any $I \subset J$. In other words J is the minimal by inclusion subcollection with the smallest defect. It is useful sometimes to define empty collection to be the unique essential subcollection in the case $d(A) = 0$.

This definition is related to the definition of an essential subcollection given in [St], but is different in general. Sturmfels was interested in resultants, so the definition in [St] was adapted to the case $d(A) = -1$ in which the definition given in [St] coincides with the one in this paper.

The essential subcollection is unique. For $d = -1$ this was shown in Sturmfels, the proof for an arbitrary $d < 0$ is the subject of the first part of next section (see Lemma 2).

3. Geometry: the defect, essential subcollections.

3.1. **Uniqueness of essential subcollection.** Let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be finite subsets of the lattice $\mathbb{Z}^n$. As before, for any $J \subset \{1, \dots k\}$ let $L(J)$ be a vector subspace parallel to the minimal affine subspace containing the Mincowski sum $\mathcal{A}_J = \sum \mathcal{A}_i$ with $i \in J$.

The defect is subadditive with respect to disjoint unions. More precisely, for disjoint $I, J \subset \{1, \dots k\}$ the following is true:

$$def(I \cup J) = def(I) + def(J) - \dim(L(I) \cap L(J)). \tag{2}$$

Lemma 1. *Let $K = I \cap J$, then $def(I \cup J) \leq def(I) + def(J) - def(K)$.*

Proof. By the definition of the defect we have:

$$def(I \cup J) = \dim(L(I \cup J)) - \#(I \cup J) = \dim(L(I \cup J)) - i - j + k,$$

where i, j, k are the sizes of I, J, K respectively. But also

$$def(I) + def(J) - def(K) = \dim(L(I)) + \dim(L(J)) - \dim(L(K)) - i - j + k,$$

so we need to compare $\dim(L(I \cup J))$ and $\dim(L(I)) + \dim(L(J)) - \dim(L(K))$. For this notice that

$$\dim(L(I \cup J)) = \dim(L(I)) + \dim(L(J)) - \dim(L(I) \cap L(J)),$$

and since $K \subset I \cap J$, the space $L(K)$ is a subspace of $L(I) \cap L(J)$, so

$$\dim(L(I \cup J)) \leq \dim(L(I)) + \dim(L(J)) - \dim(L(K)),$$

which finishes the proof. □

Corollary 2. *Let J and I be minimal by inclusion subcollections with minimal defect. Then $I \cap J = \varnothing$.*

Proof. Indeed, let $I \cap J = K \neq \varnothing$, we will show that $def(I \cup J) < def(I)$ $(= def(J))$. Since $K \subset J$ the defect of K is larger the defect of J, so $def(J) - def(K) < 0$. But by Lemma 1

$$def(I \cup J) \leq def(I) + def(J) - def(K) < def(I).$$

□

Lemma 2. *Let $\mathcal{A}_1, \dots, \mathcal{A}_k \subset \mathbb{Z}^n$ be such that the minimal defect of subcollection is non-positive, then the minimal by inclusion subcollection with minimal defect is unique.*

Proof. In the case $d(A) = 0$ the unique essential subcollection is the empty collection $J = \varnothing$.

For $d(A) < 0$, let I, and J be two minimal by inclusion subcollections with minimal defect, then by Lemma 1 $I \cap J = \varnothing$. But for disjoint subcollections I, J by relation 2 we have:

$$def(I \cup J) < def(I) + def(J) < def(I) = def(J),$$

which contradicts minimality of I and J. □

3.2. **Some properties of the essential subcollection.** Let $A = (\mathcal{A}_1, \dots, \mathcal{A}_k)$ be a collection of finite subsets of the the lattice $\mathbb{Z}^n$. For the subcollection J denote by $J^c = \{1, \dots, k\} \setminus J$ the compliment subcollection. By $\pi_J : \mathbb{R}^n \to \mathbb{R}^n / L(J)$ let us denote the natural projection to the factor space.

Lemma 3. *In the notations above the following relation holds:*

$$def(A) = def(J) + def(\pi_J(J^c)),$$

where $\pi_J(J^c)$ is the collection $(\pi_J(\mathcal{A}_i))_{i \in J^c}$.

Proof. The proof is direct calculation:

$$def(J \cup J^c) = \dim(L(J \cup J^c)) - \#(J \cup J^c) =$$
$$\dim(L(J)) + \dim L(\pi_J(J^c)) - \#(J) - \#(J^c) = def(J) + def(\pi_J(J^c)).$$

□

Corollary 3. *Let A be as be as before and let J be the unique essential subcollection of A. Then the minimal defect $d(\pi_J(J^c))$ of the projection of the complementary subcollection J^c is non-negative.*

Proof. Indeed, suppose $def(\pi_J(I)) = \dim L(\pi_J(I)) - \#I < 0$ for some $I \subset J^c$. Then by Lemma 3

$$def(J \cup I) = def(J) + def(\pi_J(I)) < def(J).$$

□

Proposition 1. *Let $A = (\mathcal{A}_1, \dots, \mathcal{A}_{n+d})$ be a collection of finite subsets of $\mathbb{Z}^n$ such that A is an essential collection of defect $-d$, i.e.*

$$-d = d(A) = def(A) < def(J),$$

for any proper $J \subset \{1, \dots, n+d\}$. Then there exists a subcollection I of size $dim(L(A)) = n$ with $d(I) = 0$.

Proof. The proof will be by induction in d. For $d = 0$ there is nothing to prove. Assume that the statement is true for any $i \le d - 1$.

Let $A = (\mathcal{A}_1, \dots, \mathcal{A}_{n+d})$ be as before, take subcollection $(\mathcal{A}_1, \dots, \mathcal{A}_{n+d-1})$, its defect $def(\mathcal{A}_1, \dots, \mathcal{A}_{n+d-1}) = -d + 1$. Let J be the essential subcollection for $(\mathcal{A}_1, \dots, \mathcal{A}_{n+d-1})$. By induction hypothesis there exists a subcollection $J' \subset J$ of size $dim(L(J)) = \#J - d + 1$ with $d(J') = 0$.

By Proposition 1 $d(\pi_J(J^c))$ is non-negative, so by Lemma 3

$$d(J' \cup J^c) \geq 0,$$

since $L(J') = L(J)$. But since

$$\#(J' \cup J^c) = \#J' + \#J^c = (\#J - d + 1) + (n + d - 1 - \#J) = n,$$

the minimal defect $d(J' \cup J^c) = 0$ and therefore $(J' \cup J^c)$ is the required subcollection. □

4. The main theorem.

In this section we will prove the main theorem. For a collection $A = (\mathcal{A}_1, \ldots, \mathcal{A}_k)$ of finite subsets of $\mathbb{Z}^n$ and subcollection J let $\mathcal{A}_J$, $L(J)$, and π_J be as before. Furthermore, denote by

- $\Lambda(J) = L(J) \cap \mathbb{Z}^n$ a lattice of integral points in $L(J)$;
- $G(J)$ the group generated by all the differences of the form $(a - b)$ with $a, b \in \mathcal{A}_i$ for any $i \in J$;
- $ind(J)$ the index of $G(\mathcal{A})$ in $\Lambda(G)$;

4.1. Zero set of the generic essential system.

4.1. Zero set of the generic essential system. In this section we will work with the systems of Laurent polynomials $f_1 = \ldots = f_k$ with supports in $A = (\mathcal{A}_1, \ldots, \mathcal{A}_k)$ such that the essential subcollection is A itself. We call such systems essential.

Theorem 5. *Let $A = (\mathcal{A}_1, \ldots, \mathcal{A}_{n+d})$ be a collection of finite subsets of $\mathbb{Z}^n$ such that $ind(A) = 1$. Let also A be an essential collection, i.e.*

$$d = d(A) = def(A) < def(J),$$

for any proper $J \subset \{1, \ldots, n+d\}$. Then for a generic consistent system $\mathbf{f} \in X_A \subset \Omega_A$ the zero set $Y_{\mathbf{f}}$ is a single point.

Here and everywhere in this paper by a generic point in the algebraic variety X we mean a point in $X \setminus \Sigma$ for the fixed subvariety Σ of smaller dimension.

Proof. By Proposition 1 there exists a subcollection I of A of size n with $d(I) = 0$. Without loss of generality let us assume that $I = \{1, \ldots, n\}$. The space Ω_A of polynomials with supports in A could be thought as a product

$$\Omega_A = \Omega_I \times \Omega_{I^c},$$

where Ω_I and Ω_{I^c} are the spaces of systems of Laurent polynomials with supports in I and I^c respectively. Let $p : \Omega_A \to \Omega_I$ be the natural projection on the first factor.

By the Bernstein criterion the subsystem $f_1 = \ldots = f_n = 0$ is generically consistent. Moreover, the Bernstein-Kouchnirenko Theorem asserts that the generic number of solutions in $(\mathbb{C}^*)^n$ is $n!Vol(\Delta_1, \ldots, \Delta_n)$, where Δ_i is the convex hull of

$\mathcal{A}_i$, and in particular is finite. Let us denote by $\Omega_I^{gen} \subset \Omega_I$ the Zariski open subset of systems $f_1 = \ldots = f_n = 0$ with exactly $n!Vol(\Delta_1, \ldots, \Delta_n)$ roots.

For each point $\mathbf{f}_I \in \Omega_I^{gen}$ the preimage $p^{-1}(\mathbf{f}_I)$ of the projection p restricted to the set of consistency X_A is a union of $n!Vol(\Delta_1, \ldots, \Delta_n)$ vector spaces $V_j(\mathbf{f}_I)$'s of dimension $|\mathcal{A}_{n+1}| + \ldots + |\mathcal{A}_{n+d}| - d$ each. Since $G(A) = \mathbb{Z}^n$ the intersection of any two of these vector spaces has smaller dimension for generic $\mathbf{f}_I \in \Omega_I^{gen}$. Denote by $X'_A \subset X_A$ the set of points which belongs to exactly one of the $V_j(\mathbf{f}_I)$'s.

By construction, the dimension of X'_A is equal to $|\mathcal{A}_1| + \ldots + |\mathcal{A}_{n+d}| - d = dim(X_A)$. Since X_A is irreducible, the complement $\Sigma = X_A \setminus X'_A$ is an algebraic subvariety of smaller dimension. But for any $\mathbf{f} \in X'_A$ the zero set $Y_\mathbf{f}$ is a single point, so the theorem is proved. □

Corollary 4. *Let $A = (\mathcal{A}_1, \ldots, \mathcal{A}_k)$ be an essential collection of finite subsets of $\mathbb{Z}^n$ of defect $d(A) = def(A) = d$. Then for the generic $\mathbf{f} \in X_A \subset \Omega_A$ the zero set $Y_\mathbf{f}$ is a finite disjoint union of $ind(A)$ subtori of dimension $n - k + d$ which are different by a multiplication by elements of $(\mathbb{C}^*)^n$.*

Proof. The lattice $G(A)$ generated by all of the differences in $\mathcal{A}_i$'s defines a torus $T \simeq (\mathbb{C}^*)^{k-d}$ for which $G(A)$ is the lattice of characters. The inclusion $G(A) \hookrightarrow \mathbb{Z}^n$ defines the homomorphism:

$$p : (\mathbb{C}^*)^n \to T.$$

The kernel of the homomorphism p is the subgroup of $(\mathbb{C}^*)^n$ consisting of finite disjoint union of $ind(A)$ subtori of dimension $n - k + d$ which are different by a multiplication by elements of $(\mathbb{C}^*)^n$.

The multiplication of Laurent polynomials by monomials does not change the zero set of a system. For any i let $\tilde{\mathcal{A}}_i$ be any translation of $\mathcal{A}_i$ belonging to $G(J)$. We can think of $\tilde{\mathcal{A}}_i$ as support of a Laurent polynomial on T. We will denote by $\tilde{A}$ the collection $(\tilde{\mathcal{A}}_1, \ldots, \tilde{\mathcal{A}}_k)$ understood as a collection of supports of Laurent polynomials on the torus T. The collection $\tilde{A}$ satisfies the assumptions of Theorem 5.

With a system $\mathbf{f} \in \Omega_A$ one can associate a system of Laurent polynomials $\tilde{\mathbf{f}}$ on T in a way described above. The zero set of $Y_\mathbf{f}$ of a system $\mathbf{f}$ is given by

$$Y_\mathbf{f} = p^{-1}(Y_{\tilde{\mathbf{f}}}) \text{ (in particular } Y_\mathbf{f} \simeq Y_{\tilde{\mathbf{f}}} \times ker(p)),$$

where $Y_{\tilde{\mathbf{f}}}$ is the zero set of a system $\tilde{\mathbf{f}}$ on T. By Theorem 5 for the generic system $\tilde{\mathbf{f}} \in X_{\tilde{A}} \subset \Omega_{\tilde{A}}$ the zero set $Y_{\tilde{\mathbf{f}}}$ which finishes the proof. □

4.2. General systems.

Theorem 6. *Let $A = (\mathcal{A}_1, \ldots, \mathcal{A}_k)$ be a collection of finite subsets of $\mathbb{Z}^n$ with the essential subcollection J. Then for the generic system $\mathbf{f} \in X_A \subset \Omega_A$ the zero set $Y_\mathbf{f}$ is a disjoint union of $ind(J)$ varieties $Y_1, \ldots, Y_{ind(J)}$ each of which is given by a Δ non-degenerate system with the same Newton polyhedra.*

Theorem 6 provides a solution for the problem of computing discrete invariants of the generic solution of the discrete invariants of generically inconsistent systems by reducing it to the classical Newton polyhedra theory. The concrete examples of applications of Theorem 6 are given in the next section.

Proof. Without loss of generality let us assume that $J = \{1, \dots, l\}$. By Corollary 4 there exists a Zariski open subset $X'_A \subset X_A$, such that for any $\mathbf{f} = (f_1, \dots, f_k) \in X'_A$ the zero set of the system $f_1 = \ldots = f_l = 0$ is a finite disjoint union of $ind(J)$ subtori $V_1, \dots, V_{ind(J)}$ which are different by a multiplication by an element of $(\mathbb{C}^*)^n$.

For the generic point $\mathbf{f} = (f_1, \dots, f_k) \in X'_A$ the restrictions of Laurent polynomials $f_{l+1}, \dots, f_k$ to each V_i are non-degenerate Laurent polynomials with Newton polyhedra $\pi_J(\Delta_{l+1}), \dots, \pi_J(\Delta_{l+1})$, respectively. □

Corollary 5. *For the generic system $\mathbf{f} \in X_A \subset \Omega_A$ the zero set $Y_\mathbf{f}$ is a non-degenerate complete intersection. That is $Y_\mathbf{f}$ is defined by $codim(Y_\mathbf{f})$ equations with independent differentials.*

Proof. Indeed, the □

5. Discrete invariants

Theorem 6 asserts allowed us to compute any discrete invariant .

Theorem 7 (Number of roots)**.** *Let $\mathcal{A}_1, \dots, \mathcal{A}_{n+k} \subset \mathbb{Z}^n$ be such that $d(\mathcal{A}) = -k$ and J be the unique essential subcollection. Then the zero set $Y_\mathbf{f}$ of the generic consistent system has dimension 0 and the number of points in $Y_\mathbf{f}$ is equal to*

$$(n - \#J + k)! \cdot ind(J) Vol(\pi_J(\Delta_i)_{i \notin J})),$$

where Δ_i is the convex hull of $\mathcal{A}_i$ and Vol is the mixed volume on $\mathbb{R}^n/L(J)$ normalized with respect to the lattice $\mathbb{Z}^n/\Lambda(J)$.

If $k = 0$ this theorem coincides with the Bernstein-Kouchnirenko theorem. In the case $k = 1$ the generic number of solution appears as the corresponding degree of A-resultant and was computed in [D'AS].

Proof. First note that for generic $\mathbf{f} \in X_\mathcal{A}$ the dimension $dim(Y_\mathbf{f})$ is equal to $dim(\tilde{X}_\mathcal{A}) - dim(X_\mathcal{A}) = 0$. By Theorem 6 the generic zero set $Y_\mathbf{f}$ is a disjoint union of $ind(J)$ varieties $Y_1, \dots, Y_{ind(J)}$ each of which is defined by generic system with newton polytopes $pi_J(\Delta_i)$ for $i \notin J$. By the Bernstein-Kouchnirenko formula the number of points in Y_i is finite and is equal to $n - \#J + k)! Vol(\pi_J(\Delta_i)_{i \notin J}))$. Therefore, the number of points in $Y_\mathbf{f}$ is

$$|Y_\mathbf{f}| = \sum_{i=1}^{ind(J)} |Y_i| = (n - \#J + k)! \cdot ind(J) Vol(\pi_J(\Delta_i)_{i \notin J})).$$

□

For simplicity, we will formulate next two results in the "hypersurface" case.

Theorem 8. *Let $\mathcal{A}_1, \dots, \mathcal{A}_k \subset \mathbb{Z}^n$ be such that $d(\mathcal{A}) < 0$ and let $J = \{2, \dots, k\}$ be the unique essential subcollection. Then the Euler characteristic and the geometric genus of the zero set $Y_{\mathbf{f}}$ of the generic consistent system is given by*

$$\chi(Y_{\mathbf{f}}) = (-1)^{n-dim(J)-1}(n - dim(J))! \cdot ind(J)Vol(\pi_J(\Delta_1)),$$
$$g(Y_{\mathbf{f}}) = ind(J)\left(B^+(\pi_J(\Delta_1))\right),$$

where Δ_1 is the convex hull of $\mathcal{A}_1$, Vol is the volume on $\mathbb{R}^n/L(J)$ normalized with respect to the lattice $\mathbb{Z}^n/\Lambda(J)$, and $B^+(\Delta)$ is the is the number of integral point in the interior of Δ.

Proof. Indeed, by Theorem 6 the generic zero set $Y_{\mathbf{f}}$ is a disjoint union of $ind(J)$ varieties $Y_1, \dots, Y_{ind(J)}$ each of which is defined by a generic equation with a Newton polytope $\pi_J(\Delta_1)$. Therefore, the Euler characteristic of Y_i is given by $\chi(Y_i) = (-1)^{n-dim(J)-1}(n - dim(J))!Vol(\pi_J(\Delta_1))$ (see [Kh1]), and the geometric genus of Y_i is given by $g(Y_i) = (B^+(\pi_J(\Delta_1)))$ (see [Kh]). The theorem follows from the additivity of the Euler characteristic and the geometric genus. ☐

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